Lecture 3 - 2003

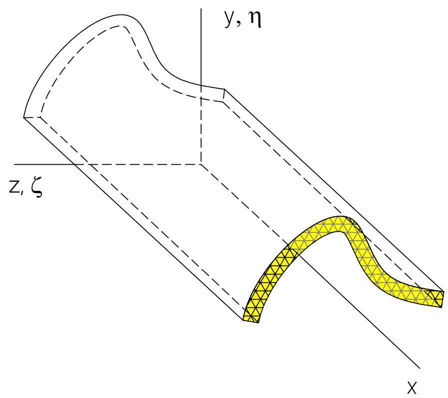
Kollbruner Section 5.2 Characteristics of Thin Walled Sections and ..
Kollbruner Section 5.3 Bending without Twist

thin walled => (cross section shape arbitrary and thickness can vary)
- axial stresses and shear stress along center of wall govern
  - normal (to curved cross section) stress neglected
- position determined by curvelinear coordinate s along center line of cross section
- St. Venant torsion not a player as K ~ t^3

use shear flow:

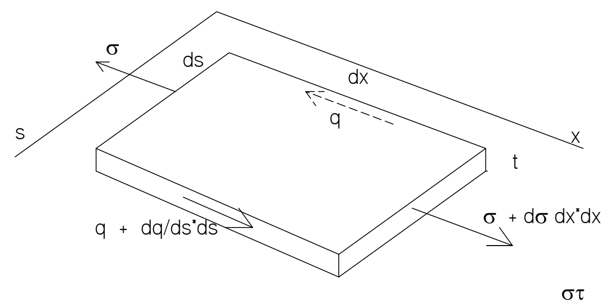
$z \Rightarrow \zeta$

$x \Rightarrow \xi$



a) equilibrium of wall element:

$\tau = \tau_{xs} = \tau_{sx}$



$$\left[\sigma + \left(\frac{d}{dx}\sigma\right)\cdot dx\right]\cdot t\cdot ds - \sigma\cdot t\cdot ds + \left(q + \frac{d}{ds}q\cdot ds\right)\cdot dx - q\cdot dx = 0$$

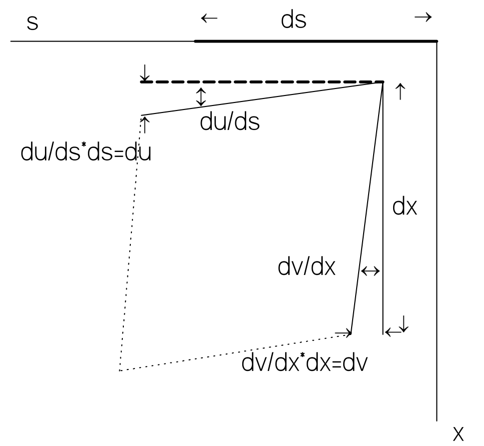
using $q = \tau\cdot t$
as it includes τ(s) and t(s)

$$\frac{d}{ds}q + \left(\frac{d}{dx}\sigma\right)\cdot t = 0$$

b) compatibility (shear strain)

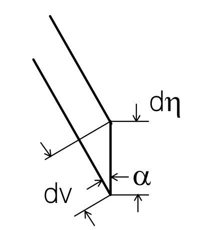
$$\frac{d}{ds}u + \frac{d}{dx}v = \gamma$$

v is displacement in s direction
u is displacement in x direction

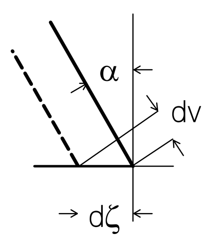


view looking ⊥ to surface

c) tangential displacement (δv) in terms of η, ζ and ϕ



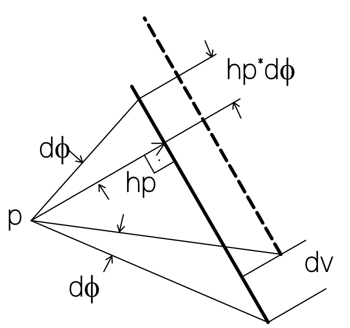
y component



z component

v is displacement in s direction
η is displacement in y direction
ζ is displacement in z direction
and ϕ is rotation
all at point s on cross section.

δ v is differential over distance dx
$\delta\eta$ is component of δv in y
$\delta\zeta$ is component of δv in z
direction
 direction
and hp*$\delta\phi$ is component due to
differential rotation between x and
x+dx



rotation component

superposition =>

$$dv = d\eta\cdot\cos(\alpha) + d\zeta\cdot\sin(\alpha) + h_p\cdot d\phi$$

η, ζ and ϕ depend on s and x while α and hp are independent of x (prismatic section)
rewrite as:

$$\frac{\delta v}{\delta x} = \frac{\delta \eta}{\delta x}\cdot\cos(\alpha) + \frac{\delta \zeta}{\delta x}\cdot\sin(\alpha) + h_p\cdot\frac{\delta \phi}{\delta x}$$

further assumptions:

1) preservation of cross section shape => $\zeta = \zeta(x)$; $\eta = \eta(x)$ $\phi = \phi(x)$

2) shear though finite is small ~ 0 => $\frac{d}{ds}u = -\left(\frac{d}{dx}v\right)$

3) Hooke's law holds => $\sigma = E\cdot\frac{\delta u}{\delta x}$ axial stress

equilibrium for the cross section:

$$\int \sigma\, dA = N_x$$

$$\int \sigma\cdot y\, dA = -M_z$$

$$\int \sigma\cdot z\, dA = M_y$$

$$\int \tau\cdot h_p\, dA = \int q\cdot h_p\, ds = T_p$$

$$\int \tau\cdot\cos(\alpha)\, dA = \int q\cdot\cos(\alpha)\, ds = V_y$$

$$\int \tau\cdot\sin(\alpha)\, dA = \int q\cdot\sin(\alpha)\, ds = V_z$$

Nx = axial force
Mz and My are bending moments wrt y and z respectively
note integral expressions
Tp is torsional moment wrt cross sectional point P
Qx and Qy shear forces

5.3 Bending without twist

$$\int \sigma\, dA = 0 \qquad \int q\cdot h_p\, ds = 0$$

possible only if lateral loads pass through P

$$\frac{\delta v}{\delta x} = \frac{\delta \eta}{\delta x}\cdot\cos(\alpha) + \frac{\delta \zeta}{\delta x}\cdot\sin(\alpha) + h_p\cdot\frac{\delta \phi}{\delta x}$$ from above,

using $\frac{d}{ds}u = -\left(\frac{d}{dx}v\right)$ with no twist => $\frac{\delta \phi}{\delta x} = 0$

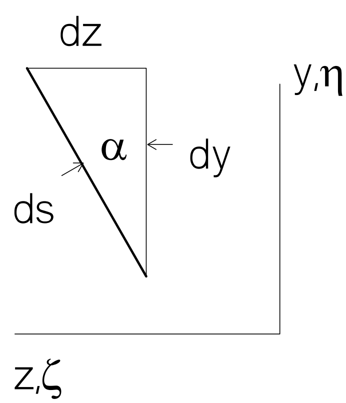
becomes: $\frac{\delta u}{\delta s} = -\frac{d\eta}{dx}\cdot\cos(\alpha) - \frac{d\zeta}{dx}\cdot\sin(\alpha)$ which can be integrated to become:

$u = -\eta'\cdot\int \cos(\alpha)\,ds - \zeta'\cdot\int \sin(\alpha)\,ds + u_0(x)$ where $\frac{d\zeta}{dx} = \zeta'$ (prime is control F7)

and $\frac{\delta \eta}{\delta x}$ is η', and comes outside the integral due to our prismatic assumption

dY (=dy) =ds*cos(α), dZ (=dz) =ds*sin(α),

where Y and Z refer to a coordinate system with an arbitrary origin, whereas x and y are defined centroidal (refer to center of area) =>



$$u(x) = -\eta'\cdot\int 1\,dY - \zeta'\cdot\int 1\,dZ + u_0(x)$$

$$u = -\eta'\cdot Y - \zeta'\cdot Z + u_0(x)$$

which says longitudinal displacement u is distributed linearly across cross section (plane sections remain plane)

u_0 is the constant of integration which is f(x)

axial strain = du/dx => $u' = -\eta''\cdot Y - -\zeta''\cdot Z + u'_0(z)$ and

$$\sigma = E\cdot u' = -E\cdot\eta''\cdot Y - \zeta''\cdot Z + E\cdot u'_0(z)$$

now with $\int \sigma\,dA = 0$

$$\int \sigma\,dA = \int E\cdot u'\,da = -\int E\cdot\eta''\cdot Y\,dA - \int E\cdot\zeta''\cdot Z\,dA + \int E\cdot u'_0(z)\,dA = 0$$ =>

$$-E\cdot\eta''\cdot\int Y\,dA - E\cdot\zeta''\cdot\int Z\,dA + E\cdot u'_0(z)\cdot\int 1\,dA = 0$$ =>

$$E\cdot u'_0(z) = E\cdot\eta''\cdot\frac{\int Y\,dA}{A} + E\cdot\zeta''\cdot\int Z\,dA$$

determines $E\cdot u'_0(z)$ in stress σ =>

$$\sigma = -E\cdot\eta''\cdot Y - -E\cdot\zeta''\cdot Z + E\cdot\eta''\cdot\frac{\int Y\,dA}{A} + E\cdot\zeta''\cdot\frac{\int Z\,dA}{A}$$

rearranged becomes

$$\sigma = -E\cdot\eta''\cdot\left(Y - \frac{\int Y\,dA}{A}\right) - -E\cdot\zeta''\cdot\left(Z - \frac{\int Z\,dA}{A}\right)$$

but ....

$\frac{\int Y\,dA}{A}$ is the definition of the y position of the centroid and $\frac{\int Z\,dA}{A}$ the z position =>

$$y = Y - \frac{\int Y\,dA}{A} \quad \text{and} \quad z = Z - \frac{\int Z\,dA}{A} \quad \text{and} \quad \sigma = -E\cdot\eta''\cdot y - E\cdot\zeta''\cdot z$$

where y and z are the position of the point s in the centroidal coordinate system.

η'' and ζ'' are determined by the equilibrium conditions

$$\int \sigma\cdot y\,dA = -M_z \qquad \int \sigma\cdot z\,dA = M_y \qquad \sigma = -E\cdot\eta''\cdot y - E\cdot\zeta''\cdot z$$

$$\int \sigma\cdot y\,dA = \int (-E\cdot\eta''\cdot y - E\cdot\zeta''\cdot z)\cdot y\,dA = -E\cdot\eta''\cdot\left(\int y\cdot y\,dA\right) \cdot -E\cdot\zeta''\cdot\left(\int z\cdot y\,dA\right) = -M_z$$

$$\int \sigma \cdot z\, dA = \int (-E\cdot\eta''\cdot y - E\cdot\zeta''\cdot z)\cdot z\, dA = -E\cdot\eta'' \int y\cdot z\, dA - E\cdot\zeta'' \int z\cdot z\, dA = M_y$$

noting that $\int z\cdot z\, dA = Iy$, $\int z\cdot y\, dA = Iyz$ and $\int y\cdot y\, dA = Iz$ and solving the two equations for two unknowns $E\cdot\eta''$ and $E\cdot\zeta''$ leads to =>

Given

$$-E\eta''\cdot I_z - E\zeta''\cdot I_{yz} = -M_z$$

$$-E\eta''\cdot I_{yz} - E\zeta''\cdot I_y = M_y$$

solving two equations two unknowns

$$\text{Find}(E\eta'', E\zeta'') \rightarrow \begin{pmatrix} \dfrac{I_{yz}\cdot M_y + M_z\cdot I_y}{-I_{yz}^{\,2} + I_y\cdot I_z} \\ \dfrac{-I_{yz}\cdot M_z - M_y\cdot I_z}{-I_{yz}^{\,2} + I_y\cdot I_z} \end{pmatrix}$$

$$E\eta'' := \frac{(I_{yz}\cdot M_y + M_z\cdot I_y)}{(-I_{yz}^{\,2} + I_y\cdot I_z)} \qquad E\zeta'' := \frac{(-I_{yz}\cdot M_z - M_y\cdot I_z)}{(-I_{yz}^{\,2} + I_y\cdot I_z)}$$

reversed signs

now substituting back into the relation for axial stress ($\sigma := -E\cdot\eta''\cdot y - E\cdot\zeta''\cdot z$) =>

$$\sigma := -E\eta''\cdot y - E\zeta''\cdot z$$

$$\sigma \rightarrow \frac{-(I_{yz}\cdot M_y + M_z\cdot I_y)}{-I_{yz}^{\,2} + I_y\cdot I_z}\cdot y - \frac{-I_{yz}\cdot M_z - M_y\cdot I_z}{-I_{yz}^{\,2} + I_y\cdot I_z}\cdot z$$

or .... rearranging in terms of My and Mz =>

$$\sigma := \frac{(-I_y\cdot y + I_{yz}\cdot z)\cdot M_z + (I_z\cdot z - I_{yz}\cdot y)\cdot M_y}{(-I_{yz}^{\,2} + I_y\cdot I_z)}$$

as a check on this development let:

$I_{yz} := 0$ and $\sigma := \frac{-(I_{yz} \cdot M_y + M_z \cdot I_y)}{(-I_{yz}^2 + I_y \cdot I_z)} \cdot y - \frac{(-I_{yz} \cdot M_z - M_y \cdot I_z)}{(-I_{yz}^2 + I_y \cdot I_z)} \cdot z$

$\sigma \rightarrow \frac{-M_z}{I_z} \cdot y + \frac{M_y}{I_y} \cdot z$ which matches our previous understanding on bending

c) Shear Stress

integration of $\frac{d}{ds} q + \left(\frac{d}{dx}\sigma\right) \cdot t = 0$ (the equilibrium relationship above) along s leads to :

$$q(s,x) = q_1(x) - \int_0^s \left(\frac{d}{dx}\sigma\right) \cdot t \, ds$$

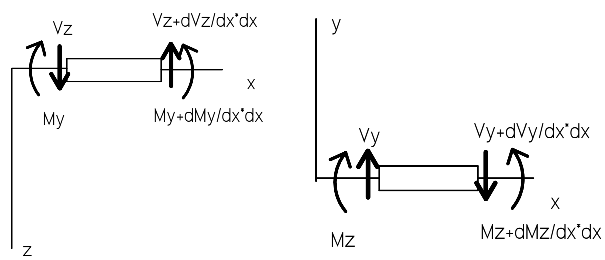
where $q_1(x)$ is f(x) and represents the shear flow at the start of the region. it is 0 at a stress free boundary which is convenient for an open section:

$q_1(x) = 0$

$$\frac{d}{dx}\sigma = \frac{d}{dx}\left[\frac{-(I_{yz} \cdot M_y + M_z \cdot I_y)}{(-I_{yz}^2 + I_y \cdot I_z)} \cdot y - \frac{(-I_{yz} \cdot M_z - M_y \cdot I_z)}{(-I_{yz}^2 + I_y \cdot I_z)} \cdot z\right]$$

where only My and Mz are x dependent and

$\frac{d}{dx} M_z = V_y$ and $\frac{d}{dx} M_y = -V_z$ =>



$M_z(x) := V_y \cdot x$ $\quad M_y(x) := -V_z \cdot x$ $\quad I_{yz} := I_{yz}$

$$\frac{d}{dx}\left[\frac{-(I_{yz} \cdot M_y(x) + M_z(x) \cdot I_y)}{(-I_{yz}^2 + I_y \cdot I_z)} \cdot y - \frac{(-I_{yz} \cdot M_z(x) - M_y(x) \cdot I_z)}{(-I_{yz}^2 + I_y \cdot I_z)} \cdot z\right] \rightarrow \frac{I_{yz} \cdot V_z - V_y \cdot I_y}{-I_{yz}^2 + I_y \cdot I_z} \cdot y - \frac{-I_{yz} \cdot V_y + V_z \cdot I_z}{-I_{yz}^2 + I_y \cdot I_z} \cdot z$$

copy and substitute

$$q(s,x) = -\int_0^s \left(\frac{d}{dx}\sigma\right) \cdot t \, ds = -\int_0^s \left[\frac{(I_{yz} \cdot V_z - V_y \cdot I_y)}{(-I_{yz}^2 + I_y \cdot I_z)} \cdot y - \frac{(-I_{yz} \cdot V_y + V_z \cdot I_z)}{(-I_{yz}^2 + I_y \cdot I_z)} \cdot z\right] \cdot t \, ds$$

$$q(s,x) = \frac{-1}{-I_{yz}^2 + I_y \cdot I_z} \cdot \left[(I_{yz} \cdot V_z - V_y \cdot I_y) \cdot \int_0^s y \cdot t \, ds - (-I_{yz} \cdot V_y + V_z \cdot I_z) \cdot \int_0^s z \cdot t \, ds\right]$$

if we designate the integrals which are the static moments of the cross section area: Qy and Qz:

$$Q_z = \int_0^s y \cdot t\, ds \qquad Q_y = \int_0^s z \cdot t\, ds$$

$$q(s,x) := \frac{-1}{-I_{yz}^2 + I_y \cdot I_z} \cdot \left[\left(I_{yz} \cdot V_z - V_y \cdot I_y\right) \cdot Q_z - \left(-I_{yz} \cdot V_y + V_z \cdot I_z\right) \cdot Q_y \right]$$

or rearranging as we did for axial stress

$$q(s,x) \text{ collect}, V_y, V_z \rightarrow \frac{-1}{-I_{yz}^2 + I_y \cdot I_z} \cdot \left(-I_y \cdot Q_z + I_{yz} \cdot Q_y\right) \cdot V_y - \frac{1}{-I_{yz}^2 + I_y \cdot I_z} \cdot \left(I_{yz} \cdot Q_z - I_z \cdot Q_y\right) \cdot V_z$$

$$q(s,x) := \frac{-1}{\left(-I_{yz}^2 + I_y \cdot I_z\right)} \cdot \left[\left(I_{yz} \cdot Q_z - I_z \cdot Q_y\right) \cdot V_z + \left(-I_y \cdot Q_z + I_{yz} \cdot Q_y\right) \cdot V_y \right]$$

or if the axes are principal (Ixy = 0)

$$I_{yz} := 0$$

$$q(s,x) := \frac{-1}{\left(-I_{yz}^2 + I_y \cdot I_z\right)} \cdot \left[\left(I_{yz} \cdot Q_z - I_z \cdot Q_y\right) \cdot V_z + \left(-I_y \cdot Q_z + I_{yz} \cdot Q_y\right) \cdot V_y \right]$$

$$q(s,x) \text{ simplify collect }, Q_y \rightarrow \frac{V_z}{I_y} \cdot Q_y + Q_z \cdot \frac{V_y}{I_z} \qquad q(s,x) := \left(\frac{Q_y \cdot V_z}{I_y} + \frac{Q_z \cdot V_y}{I_z} \right) \qquad I_{yz} := 0$$

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d) Shear Center

the above relationships apply for bending without twist i.e. when

$$\int \sigma\, dA = 0 \qquad \int q \cdot h_p\, ds = 0$$

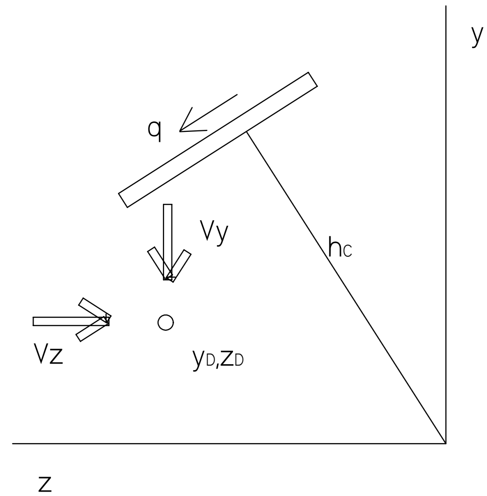
possible only if lateral loads pass through P

$$q \rightarrow \frac{-1}{-I_{zy}^2 + I_y \cdot I_z} \cdot \left[\left(I_{zy} \cdot Q_z - I_z \cdot Q_y\right) \cdot V_z + \left(-I_y \cdot Q_z + I_{zy} \cdot Q_y\right) \cdot V_y \right]$$

from above (reset in hidden area

this point P was designated (by Maillart in 1921, see page 106 Kollbrunner) as the shear center. In the centroidal coordinate system, this is located at y_D and z_D. the second condition applies for a shear force V. The center of action must pass through P. Divide Q into components Vy and Vz

thus from (Qy), the moment the moment equilibrium wrt C (in centroidal coordinates) is:



$$\int q(V_y)\cdot h_c\, ds + V_y\cdot z_D = 0$$

$$\int q(V_y)\cdot h_c\, ds = -V_y\cdot z_D$$

$q(V_y)$ is q with Vz set to 0 and hc is perpendicular distance from centroid to line of action (i.e. y(s) and z(s)).

set $V_z := 0$ reset $q := \frac{-1}{\left(-I_{yz}^2 + I_y\cdot I_z\right)}\cdot\left[\left(I_{yz}\cdot Q_z - I_z\cdot Q_y\right)\cdot V_z + \left(-I_y\cdot Q_z + I_{yz}\cdot Q_y\right)\cdot V_y\right]$

$$q \rightarrow \frac{-1}{-I_{zy}^2 + I_y\cdot I_z}\cdot\left(-I_y\cdot Q_z + I_{zy}\cdot Q_y\right)\cdot V_y$$

substitute

$$\int q(V_y)\cdot h_c\, ds = -V_y\cdot z_D = \int \frac{-1}{\left(-I_{zy}^2 + I_y\cdot I_z\right)}\cdot\left(-I_y\cdot Q_z + I_{zy}\cdot Q_y\right)\cdot V_y\cdot h_c\, ds$$

$$-V_y\cdot z_D = \frac{-V_y}{\left(-I_{zy}^2 + I_y\cdot I_z\right)}\left[\int \left(-I_y\cdot Q_z + I_{zy}\cdot Q_y\right)\cdot h_c\, ds\right]$$

or ...

$$z_D = \frac{1}{\left(-I_{zy}^2 + I_y\cdot I_z\right)}\cdot\left[\int \left(-I_y\cdot Q_z + I_{zy}\cdot Q_y\right)\cdot h_c\, ds\right] = \frac{-I_y\cdot\int Q_z\cdot h_c\, ds + I_{zy}\cdot\int Q_y\cdot h_c\, ds}{-I_{zy}^2 + I_y\cdot I_z}$$

moment from V.z $\int q(V_z)\cdot h_c\, ds - V_z\cdot y_D = 0$ $\int q(V_z)\cdot h_c\, ds = V_z\cdot y_D$

$q(V_z)$ is q with Vy set to 0 and hc is perpendicular distance from centroid to line of action (i.e. y(s) and z(s)).

reset

reset $V_z := V_z$ set $V_y := 0$ $q := \frac{-1}{\left(-I_{yz}^{2} + I_y \cdot I_z\right)} \cdot \left[\left(I_{yz} \cdot Q_z - I_z \cdot Q_y\right) \cdot V_z + \left(-I_y \cdot Q_z + I_{yz} \cdot Q_y\right) \cdot V_y\right]$

$$q \rightarrow \frac{-1}{-I_{zy}^{2} + I_y \cdot I_z} \cdot \left(I_{zy} \cdot Q_z - I_z \cdot Q_y\right) \cdot V_z$$ substitute

$$\int q(V_z) \cdot h_c \, ds = V_z \cdot y_D = \int \frac{-1}{\left(-I_{zy}^{2} + I_y \cdot I_z\right)} \cdot \left(I_{zy} \cdot Q_z - I_z \cdot Q_y\right) \cdot V_z \cdot h_c \, ds$$

$$V_z \cdot y_D = \int \frac{-1}{\left(-I_{zy}^{2} + I_y \cdot I_z\right)} \cdot \left(I_{zy} \cdot Q_z - I_z \cdot Q_y\right) \cdot V_z \cdot h_c \, ds = \frac{-V_z}{\left(-I_{zy}^{2} + I_y \cdot I_z\right)} \cdot \left(I_{zy} \cdot \int Q_z \cdot h_c \, ds - I_z \cdot \int Q_y \cdot h_c \, ds\right)$$

$$y_D = \frac{-1}{\left(-I_{zy}^{2} + I_y \cdot I_z\right)} \cdot \left(I_{zy} \cdot \int Q_z \cdot h_c \, ds - I_z \cdot \int Q_y \cdot h_c \, ds\right)$$

for principal axes (Iyz = 0): $I_{zy} := 0$

$$y_D := \frac{-1}{\left(-I_{zy}^{2} + I_y \cdot I_z\right)} \cdot \left(I_{zy} \cdot \int Q_z \cdot h_c \, ds - I_z \cdot \int Q_y \cdot h_c \, ds\right)$$

$$z_D := \frac{-I_y \cdot \int Q_z \cdot h_c \, ds + I_{zy} \cdot \int Q_y \cdot h_c \, ds}{-I_{zy}^{2} + I_y \cdot I_z}$$

$$y_D = \frac{\int Q_y \cdot h_c \, ds}{I_y}$$

$$z_D = \frac{-\int Q_z \cdot h_c \, ds}{I_z}$$