2.092/2.093 — Finite Element Analysis of Solids & Fluids I Fall '09

Lecture 3 - Analysis of Solids/Structures and Fluids

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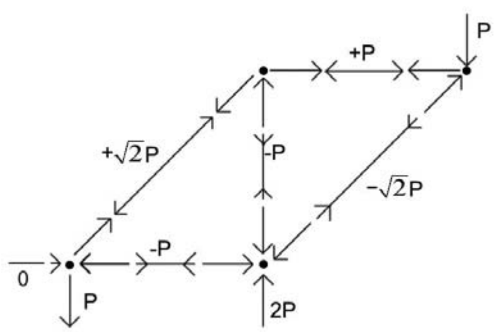
The fundamental conditions to be satisfied are:

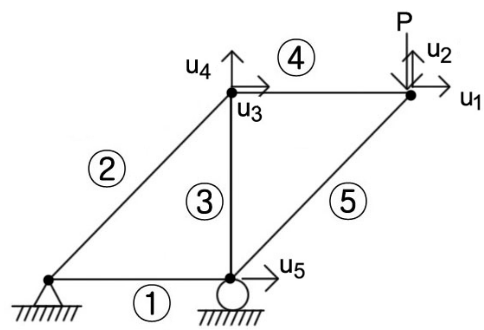
I. Equilibrium: in solids, $\boldsymbol{F} = m\boldsymbol{a}$; in fluids, conservation of momentum

II. Compatibility: continuity and boundary conditions

III. Constitutive relations: Stress/strain law

Each joint and each element must be in equilibrium. From last lecture:





We want to solve $\boldsymbol{KU} = \boldsymbol{R}$ for this system. We know that

$$\boldsymbol{R} = \begin{bmatrix} 0 \\ -P \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

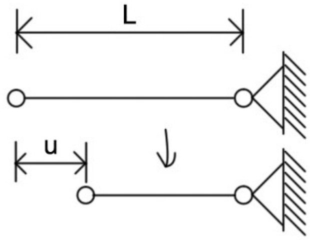
To solve for $\boldsymbol{K}$, assume $u_5 = 1$, $u_1 = u_2 = u_3 = u_4 = 0$. Then, the left hand side becomes

$$\begin{bmatrix} k_{15} \\ k_{25} \\ k_{35} \\ k_{45} \\ k_{55} \end{bmatrix} = \boldsymbol{R}$$

where $\boldsymbol{R}$ are the external applied forces corresponding to the imposed displacement.

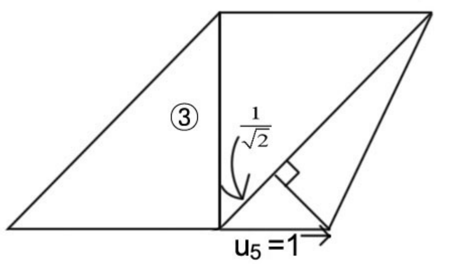
Example

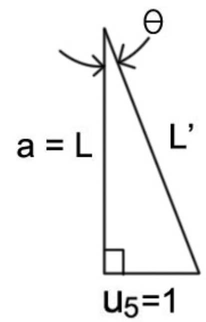
Consider a simple bar:



$$F = \frac{EA}{L} \times [\text{Displacement u}] \quad \rightarrow \quad \tilde{K} = \frac{EA}{L}$$

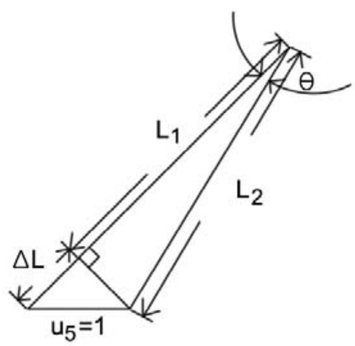
Truss bars can only resist axial forces. Note that the length of bar 3 does not change in infinitesimal displacement analysis!





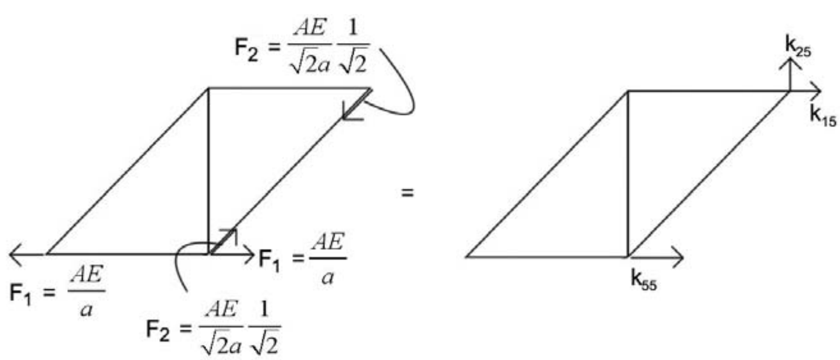
$$L' \cos\theta = a$$

We use the approximations $\cos\theta = 1$, $\sin\theta = \theta$ because the displacements are small, and we are performing an infinitesimal displacement analysis. So, $L' = a$.



$$L_1 = L_2$$

$$\Delta L = \frac{1}{\sqrt{2}}$$



$$k_{55} = \frac{AE}{2\sqrt{2}a} + \frac{AE}{a} \quad ; \quad k_{15} = -\frac{AE}{2\sqrt{2}a} \quad ; \quad k_{25} = -\frac{AE}{2\sqrt{2}a}$$

$$\boldsymbol{K} = \frac{AE}{a}\begin{bmatrix} . & . & . & . & -\frac{1}{2\sqrt{2}} \\ . & . & . & . & -\frac{1}{2\sqrt{2}} \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & . & \frac{1}{2\sqrt{2}}+1 \end{bmatrix}$$

Using this method, we can construct the whole stiffness matrix with the displacement boundary conditions removed. Our $\boldsymbol{KU}=\boldsymbol{R}$ becomes

$$\begin{bmatrix} \underset{5\times5}{\boldsymbol{K}_{aa}} & \underset{5\times3}{\boldsymbol{K}_{ab}} \\ \underset{3\times5}{\boldsymbol{K}_{ba}} & \underset{3\times3}{\boldsymbol{K}_{bb}} \end{bmatrix}\begin{bmatrix} \boldsymbol{u}_a \\ \boldsymbol{u}_b \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_a \\ \boldsymbol{R}_b \end{bmatrix}$$

$$\boldsymbol{u}_a = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \text{ and } \quad \boldsymbol{u}_b = \begin{bmatrix} u_6 \\ u_7 \\ u_8 \end{bmatrix} \quad ; \quad \boldsymbol{R}_a = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{bmatrix} \text{ and } \boldsymbol{R}_b = \begin{bmatrix} R_6 \\ R_7 \\ R_8 \end{bmatrix}$$

Now we have the simplified equation $\boldsymbol{K}_{aa}\boldsymbol{U}_a = \boldsymbol{R}_a$. Solve for $\boldsymbol{U}_a$, and then the reactions are $\boldsymbol{R}_b = \boldsymbol{K}_{ba}\boldsymbol{U}_a$.

Also note: "Linear analysis" means that for any constants α, β,

$$\boldsymbol{K}\boldsymbol{U}_1 = \boldsymbol{R}_1,\ \boldsymbol{K}\boldsymbol{U}_2 = \boldsymbol{R}_2 \rightarrow \boldsymbol{K}\left(\alpha\boldsymbol{U}_1 + \beta\boldsymbol{U}_2\right) = \alpha\boldsymbol{R}_1 + \beta\boldsymbol{R}_2$$

To see why the solutions in linear analysis are unique, please see p. 239 in the textbook: Bathe, K.J. *Finite Element Procedures.* Cambridge, MA: Klaus-Jürgen Bathe, 2007. ISBN: 978-0979004902.