Given the system geometry \((V, S_u, S_f)\), loads \((f^B, f^{S_f})\), and material laws, we calculate:

- Displacements \(u, v, w\) (or \(u_1, u_2, u_3\))
- Strains, stresses

We will perform a linear elastic analysis for solids. We want to obtain the equation \(KU = R\). Recall our truss example. There, we had element stiffness \(\frac{AE}{L}\). To calculate the stiffnesses, we could proceed this way:
Every differential element should satisfy $EA \frac{d^2 u}{dx^2} = 0$. To obtain $F$, we solve:

$$EA \frac{d^2 u}{dx^2} = 0 \quad ; \quad u \bigg|_{x=0} = 1.0 \quad ; \quad u \bigg|_{x=L} = 0$$

Consider a 2D analysis:

In this case, the method used for the truss problem to get the stiffness matrix $K$ would not work. In general 3D analysis, we must satisfy (for the exact solution)

- **Equilibrium:**
  
  I. $\tau_{ij,j} + f_i^B = 0$ in $V(i, j = 1, 2, 3)$, where $\tau_{ij}$ are the Cauchy stresses (forces per unit area in the deformed geometry).

  II. $\tau_{ij} n_j = f_i^S$ on $S_f$

- **Compatibility:** $u_i = u_i^S$ on $S_u$ and all displacements must be continuous.

- **Stress-strain laws**

This is known as the differential formulation.

**Example**

Reading assignment: Section 3.3.4
- Compatibility

\[ u \bigg|_{x=0} = 0 \]  

(c)

- Stress-strain law

\[ \tau_{xx} = E \frac{du}{dx} \]  

(d)

In a 1D problem, nodes are surfaces.

In a 2D problem, we define line \( \times \) thickness = surface, but one point can belong to both \( S_f \) and \( S_u \).

**Principle of Virtual Work (Virtual Displacements)**

Clearly, the exact solution \( u(x) \) must satisfy:

\[
\left( EA \frac{d^2 u}{dx^2} + f^B \right) \delta u(x) = 0
\]

(1)

where \( \delta u(x) \) is continuous and zero at \( x = 0 \). Otherwise, it is an arbitrary function. Hence, also,

\[
\int_0^L \left( EA \frac{d^2 u}{dx^2} + f^B \right) \delta u(x) dx = 0
\]

(2)
Lecture 4 The Principle of Virtual Work 2.092/2.093, Fall ’09

From Eq. (2):

\[
EA \frac{du}{dx} \delta u \bigg|_0^L - \int_0^L \frac{d\delta u}{dx} EA \frac{du}{dx} dx + \int_0^L f^B \delta u dx = 0 \quad \text{(A)}
\]

The equation above becomes:

\[
\text{Internal virtual work } \int_0^L \frac{d\delta u}{dx} EA \frac{du}{dx} dx = \text{External virtual work } \int_0^L f^B \delta u dx + \text{Virtual work due to boundary forces } R\delta u \bigg|_L
\]

where \( \frac{d\delta u}{dx} \) are the virtual strains, \( \frac{du}{dx} \) are the real strains, and \( \delta u \) are the virtual displacements. We set \( \delta u = 0 \) on \( S_u \), since we do not know the external forces on \( S_u \). To solve \( EA \frac{d^2 u}{dx^2} + f^B = 0 \), we look for a function \( u \) where \( \frac{d^2 u}{dx^2} \) exists (\( \frac{du}{dx} \) should be continuous). In order to calculate the virtual work, we look for the solutions where only \( u \) is continuous.

Physically,

\[
\int_0^L \varepsilon_{xx} EA \varepsilon_{xx} dx = \int_0^L \bar{\pi} f^B dx + R\bar{\pi}_L
\]

(\( \text{the bar denotes ‘virtual’ quantities} \))

In 3D vector form, the principle of virtual work now becomes

\[
\int_V \varepsilon^T C \varepsilon dV = \int_V \bar{\Pi}^T f^B dV + \int_{S_f} \bar{u}^{S_f^T} \bar{f}^S dS_f \quad \text{(B)}
\]

where

\[
\varepsilon = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \quad ; \quad \varepsilon_{xx} = \frac{\partial u}{\partial x}
\]

\[
\bar{\pi} = \begin{bmatrix} \bar{\varepsilon}_{xx} \\ \bar{\varepsilon}_{yy} \\ \bar{\varepsilon}_{zz} \\ \bar{\gamma}_{xy} \\ \bar{\gamma}_{yz} \\ \bar{\gamma}_{zx} \end{bmatrix} \quad ; \quad \bar{\varepsilon}_{zz} = \frac{\partial \bar{\pi}}{\partial z}
\]

We see that (B) is the generalized form of (A’). The principle of virtual work states that for any compatible virtual displacement field imposed on the body in its state of equilibrium, the total internal virtual work is
equal to the total external virtual work. Note that this variational formulation is equivalent to the differential formulation, given earlier.