2.094
FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS
SPRING 2008

Homework 1 - Solution

Instructor:  Prof. K. J. Bathe
Assigned:  02/07/2008
Due:  02/14/2008

Problem 1 (10 points):

In this problem, the principle of virtual work reduces to

\[
\int_Y \mathbf{t} \mathbf{\tau}_{22} \mathbf{\varepsilon}_{22} d \mathbf{\psi} = \int_{S_f} \mathbf{u}_{2}^{S_f} \mathbf{t} \mathbf{f}_{x_2} d \mathbf{S}_f
\]

(a) simple tension in the \(x_1\) direction; (b) a simple tension in the \(x_2\) direction; (c) a simple shear

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**Figure 1.** Three simple independent virtual displacement patterns.

(a) a simple tension in the \(x_1\) direction; (b) a simple tension in the \(x_2\) direction; (c) a simple shear
(a) A simple extension in $x_1$ direction; $\bar{u}_1 = (t \cdot x_1 - 1)$ and $\bar{u}_2 = 0$

\[ \bar{e}_{22} = 0 \]
\[ \bar{u}_2^{sf} = 0 \]

Therefore,
\[ \int_{tV} (t \cdot e_{22} \cdot \bar{e}_{22}) \cdot d^t V = 0 \]
\[ \int_{tSf} \bar{u}_2^{sf} \cdot (t \cdot f_{x_2} \cdot d^t S_f) = 0 \]

Check!

(b) A simple extension in $x_2$ direction; $\bar{u}_1 = 0$ and $\bar{u}_2 = (t \cdot x_2 + 1)$

\[ \bar{e}_{22} = 1 \]
\[ \bar{u}_2^{sf} = 4 \text{ (on the top surface)} \text{ and } \bar{u}_2^{sf} = 0 \text{ (on the bottom surface)} \]

Therefore,
\[ \int_{tV} (t \cdot e_{22} \cdot \bar{e}_{22}) \cdot d^t V = \int_{tV} 20 \times 1 \cdot d^t V = 20 \times d^t V = 80 \]
\[ \int_{tSf} \bar{u}_2^{sf} \cdot (t \cdot f_{x_2} \cdot d^t S_f) = \int_{tSf} 4 \times 20 \cdot d^t S_f = 80 \times d^t S_f = 80 \]

Check!

(c) A simple shear; $\bar{u}_1 = 0$ and $\bar{u}_2 = (1 - t \cdot x_1)$

\[ \bar{e}_{22} = 0 \]
\[ \bar{u}_2^{sf} = 1 - t \cdot x_1 \]

Therefore,
\[ \int_{tV} (t \cdot e_{22} \cdot \bar{e}_{22}) \cdot d^t V = 0 \]
\[ \int_{tSf} \bar{u}_2^{sf} \cdot (t \cdot f_{x_2} \cdot d^t S_f) = \int_{tSf_1} 20 \times (1 - t \cdot x_1) \cdot d^t S_{f_1} + \int_{tSf_2} (-20) \times (1 - t \cdot x_1) \cdot d^t S_{f_2} \]
\[ = 20 \int_0^1 (1 - t \cdot x_1) \cdot d^t x_1 - 20 \int_0^1 (1 - t \cdot x_1) \cdot d^t x_1 = 0 \]

(on the top surface) \text{ and } \int_0^1 (1 - t \cdot x_1) \cdot d^t x_1 = 0 \text{ (on the bottom surface)}

Check!
Problem 2 (20 points):

In this solution, the horizontal symmetry line corresponds to the line CD and the vertical symmetry line corresponds to the line AB. The distance of the horizontal line is measured from the point C and the distance of the vertical line is measured from the point A. (See Figure 2.)

The stresses are shown through Figures 3 and 6. We can see that the solutions obtained with 9-node elements are better than those obtained with 4-node elements. One possible way, in general, to check calculated solutions is to see whether the stress boundary conditions are satisfied. Here, we know that we should have \( \tau_{zz} \big|_A = 25 \) due to the applied traction and \( \tau_{zz} \big|_B = \tau_{yy} \big|_D = 0 \) because no tractions are applied at the points B, C, and D. The solutions obtained with 9-node elements are closer to these exact values than those obtained with 4-node elements. In this problem we also have the analytical solutions for the \( \tau_{zz} \) at point C and D, see reference 1. The solutions are compared in Figure 7.

Figure 4. Stresses on the horizontal symmetry line solved using 9-node elements

Figure 5. Stresses on the vertical symmetry line solved using 4-node elements

Figure 6. Stresses on the vertical symmetry line solved using 9-node elements
Figure 7. $\tau_{zz}$ on the horizontal symmetry line