Solution of Problem Set 5: The Discrete Fourier Transform

Problem 1:

Given a signal of duration $T = 0.128$ ms, sampled at a rate of $F_s = 8$ kHz, the number of samples is $L = T f_s = 1024$. If only 256 samples are taken, the frequency spacing in the computed DFT is $\Delta f = f_s/N = 31.25$ Hz.

(a) The number of multiplications required for the direct computation of the DFT is $N^2 = 256^2 = 65,536$.

(b) The number of multiplications required for the computation of the DFT using the FFT is $N/2 \log_2(N) = 128 \times 8 = 1024$.

Note that $F_m = F^*(j \frac{2\pi m}{N\Delta T}) = F^*(j2\pi f_s \frac{m}{N})$. Hence, the $m^{th}$ component of the DFT corresponds to the actual frequency of $m f_s = m \Delta f = 31.25m$ Hz.

Comparing above formulas, we can realize that, if there is a ONE-TO-ONE correspondence between discrete values of $F_m$ and continuous values of $F(j\Omega)$, then at $\Omega = 2\pi f_s \frac{m}{N}$ the $F(j\Omega)$ is a delta function with an amplitude equal to $\frac{2\pi}{N} F_m$. This can be verified for example for either a DC signal or a sine signal without spectral leakage.

Note that to find the scale factor, we cannot use the relation between $F_m$ and $F^*$ ($F^*(j\Omega) = \frac{1}{\Delta T} \sum_{n=-\infty}^{n=+\infty} F(j(\Omega - n2\pi f_s))$). That’s because in relating $F_m$ to $F^*$, we assumed that the rest of the sample points are zero. On the other hand, DFT assumes that the $f(t)$ is a periodic extension of the sampled data with period $N\Delta T$.

Problem 2:

A $f_c = 10$ kHz sinusoidal signal is sampled at $f_s = 80$ kHz, and a total of $N = 64$ samples are taken to compute the DFT of the signal. The period of the signal is $T = 1/10^4 = 0.1$ ms, and the sampling period is $\Delta T = 1/f_s = 1/(80 \cdot 10^3) = 12.5 \mu$sec. Then 64 samples will contain 8 whole periods of the signal (64 $\cdot$ 12.5 $\mu$sec $\cdot$ 10kHz = 8 cycle). Therefore, there is no spectral leakage. The frequency spacing of the DFT would be $\Delta f = 1/(N\Delta T) = 1/(64 \cdot 12.5 \cdot 10^{-6}) = 1250$ Hz, so we would expect to see one peak in the DFT $\{F_m\}$ at $m_1 = f_s/f_c = 8$ and the other peak at $m_2 = N - m_1 = 56$. The following MATLAB verifies this.

```matlab
n=[0:63];
DT=1/(80*1e3);
f=sin(2*pi*DT*10000*n);
stem(n,abs(fft(f)))
```
Problem 3:

We note that

\[ W_8^0 = 1, \quad W_8^1 = \sqrt{2}/2 - j \sqrt{2}/2, \quad W_8^2 = 0 - j1, \quad W_8^3 = -\sqrt{2}/2 - j \sqrt{2}/2, \]
\[ W_8^4 = -1, \quad W_8^5 = -\sqrt{2}/2 + j \sqrt{2}/2, \quad W_8^6 = 0 + j1, \quad W_8^7 = \sqrt{2}/2 + j \sqrt{2}/2. \]

and from Fig. 5 in the FFT handout:
we can construct the following table:

<table>
<thead>
<tr>
<th>Data</th>
<th>2-point DFTs</th>
<th>4-point DFTs</th>
<th>8-point DFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0 = 5$</td>
<td>$x_0 + x_4 = 10$</td>
<td>$10 + W_8^0 \cdot (-6) = 4$</td>
<td>$4 + W_8^0 \cdot (-4) = 0 = X_0$</td>
</tr>
<tr>
<td>$x_4 = 5$</td>
<td>$x_0 - x_4 = 0$</td>
<td>$0 + W_8^0 \cdot 0 = 0$</td>
<td>$0 + W_8^0 \cdot 0 = 0 = X_1$</td>
</tr>
<tr>
<td>$x_2 = -3$</td>
<td>$x_2 + x_6 = -6$</td>
<td>$10 + W_8^4 \cdot (-6) = 16$</td>
<td>$16 + W_8^2 \cdot 0 = 16 = X_2$</td>
</tr>
<tr>
<td>$x_6 = -3$</td>
<td>$x_2 - x_6 = 0$</td>
<td>$0 + W_8^6 \cdot (0) = 0$</td>
<td>$0 + W_8^3 \cdot (0) = 0 = X_3$</td>
</tr>
<tr>
<td>$x_1 = -1$</td>
<td>$x_1 + x_5 = -2$</td>
<td>$-2 + W_8^0 \cdot (-2) = -4$</td>
<td>$4 + W_8^3 \cdot 0 = 8 = X_4$</td>
</tr>
<tr>
<td>$x_5 = -1$</td>
<td>$x_1 - x_5 = 0$</td>
<td>$0 + W_8^2 \cdot 0 = 0$</td>
<td>$0 + W_8^5 \cdot 0 = 0 = X_5$</td>
</tr>
<tr>
<td>$x_3 = -1$</td>
<td>$x_3 + x_7 = -2$</td>
<td>$-2 + W_8^1 \cdot (-2) = 0$</td>
<td>$16 + W_8^6 \cdot 0 = 16 = X_6$</td>
</tr>
<tr>
<td>$x_7 = -1$</td>
<td>$x_3 - x_7 = 0$</td>
<td>$0 + W_8^6 \cdot (0) = 0$</td>
<td>$0 + W_8^5 \cdot 0 = 0 = X_7$</td>
</tr>
</tbody>
</table>

The same answers are obtained using the Matlab command: `fft([5 -1 -3 -1 5 -1 -3 -1])`.

Assuming the samples are of the noisy signal created using the following Matlab commands:

```matlab
>> t=[0:.01:10.23]; 1024 Points
>> f=exp(-t).*sin(10*t); % Clean signal
>> noise=rand(1,1024)-0.5; % Additive random noise
>> signal=f+noise; % Additive random noise
```

Please note that we have changed the noise, such that its average value is equal to zero. Otherwise, we can subtract the DC value manually from DFT component. The DC magnitude is equal to $Nn_0$ where $N$ is the number of points and $n_0$ is the average value of Noise.

Plots of the clean and noisy signals are:
The DFT magnitude plots are

The following is the MATLAB script that was used to do crude FFT-based low-pass filtering on the data. The FFT of the data set is computed and K terms \( F_m, |m| = 0, \ldots, K \) are retained before taking the IFFT and plotting the result.
t=[0:.01:10.23];
signal = exp(-t).*sin(10*t);
noise = rand(1,1024)-0.5;
s_plus_n = signal+noise;
m = [0:1023];
Fm = fft(s_plus_n);
% Do low-pass filtering by zeroing out the center of
% the DFT. K is the band-limit
K = 60;
% Create an empty array and move the K low frequency
% components in.
Fout=zeros(size(Fm));
Fout(1:K) = Fm(1:K);
Fout(1024-K-1:1024) = Fm(1024-K-1:1024);
subplot(2,1,1), plot(m,abs(Fout))
subplot(2,1,2), plot(t,signal,'r--',t,real(ifft(Fout)),'b')

Results for some typical values of $K$ are:

(a) $K = 100$:  

![Truncated DFT Magnitude](image1)

![Original signal and Filtered s+n](image2)
(b) $K = 60$:

![Graph showing truncated DFT magnitude for $K = 60$.]

(c) $K = 40$:

![Graph showing truncated DFT magnitude for $K = 40$.]
(d) $K = 20$:

We can argue about it, but I think that $K = 40$ looks to be about a good compromise between residual noise and fidelity of the original signal???