Notes:

- The quiz is open-book.
- The time allowed is ninety minutes.
- There are six problems, answer them all.
- Partial credit will be given.
Problem 1: (15 points)
Show that to interpolate a value to the mid-point of a sampling interval in the sample set \( \{f_n\} \), \( n = 0 \ldots N - 1 \), where \( f_n = f(n\Delta T) \), the cardinal (Whittaker) reconstructor may be written
\[
f(n\Delta T + \Delta T/2) = \frac{1}{\pi} \sum_{k=0}^{N-1} f_k \frac{(-1)^k}{(n - k + 1/2)}
\]

Problem 2: (20 points)
(a) Design (find the transfer function of) a unity-gain 3rd-order low-pass Butterworth analog filter with a -3dB cut-off frequency of 10 rad/s.
(b) Make a pole-zero plot for your filter.
(c) Convert your design to a high-pass filter with the same cut-off frequency.
(d) If (do not do it) you were to convert your design to a band-pass filter, what would be the order of the filter, how many zeros would be created, and where in the s-plane would the zeros lie?

Problem 3: (15 points)
Many real-life signal processing problems involve waveforms containing echoes, or reverberation. Consider a continuous-time linear filter with impulse response
\[
h(t) = \delta(t) + \delta(t - \tau)
\]
Assume that the filter is excited with waveform \( f(t) \), and that the output is \( y(t) \)

(i) Show that the output contains an echo, that is
\[
y(t) = f(t) + f(t - \tau)
\]

(ii) Find the frequency response \( H(j\omega) \) for the filter, and express it in terms of its magnitude \( |H(j\omega)| \), and phase \( \angle H(j\omega) \).
\[\text{Hint:} \quad 1 + e^{-j\theta} = e^{-j\theta/2} \left[ e^{j\theta/2} + e^{-j\theta/2} \right] \]

(iii) Assume \( f(t) = \sin \omega t \). At what values of \( \omega \) will the filter exhibit no steady-state output?
Problem 4: (20 points)

An AM (amplitude-modulated) radio signal $f_{AM}(t)$ is described by

$$f_{AM}(t) = (1 + a f_{audio}(t)) \sin(\omega_c t)$$

where $f_{audio}(t)$ is the audio signal, $\sin(\omega_c t)$ is known as radio-frequency carrier signal ($f_c = 500 – 1600$ kHz - the AM band), and $a$ is a positive constant that determines the modulation depth. (Note that we require $|a f_{audio}(t)| < 1$ otherwise we have over-modulation.)

The following figure shows an AM signal with an “audio” waveform that is a simple low frequency sinusoid. You can see how the audio signal “modulates” the amplitude of the rf signal.

(a) Sketch the magnitude of the Fourier transform of $f_{AM}(t)$ when $f_{audio}(t) = 0$.

(b) Let $a = 0.5$, and sketch the magnitude of the Fourier transform of $f_{AM}(t)$ when

$$f_{audio}(t) = 0.5 \cos(2\pi \cdot 1000t) + 0.25 \cos(2\pi \cdot 2000t)$$

(Hint: There is no need to actually compute the FT. Consider expanding $f_{AM}(t)$, or simply use properties of the FT.)

(c) Use your result from (b) to generalize, and sketch the magnitude spectrum of $f_{AM}(t)$ when $f_{audio}(t)$ has a spectrum (again let $a = 0.5$):

(d) If $f_{audio}$ has a bandwidth $B = \omega_u - \omega_l$, what is the bandwidth of a band-pass filter that would be necessary to select the signal $f_{AM}(t)$ out of all the other AM radio stations?
Problem 5: (15 points)
The most commonly used (approximate) data reconstructor is the “zero-order hold” (ZOH) which simply “holds” the output value $y(n\Delta T)$ as a constant over the reconstruction interval:

$$y(t) = f(n\Delta T) \quad n\Delta T \leq t < (n + 1)\Delta T$$

where $f(n\Delta T)$ is the value of input at time $t = n\Delta T$. The output of the ZOH therefore looks like a staircase function, with discrete jumps to a new value at each update time $n\Delta T$:

(a) Show that the ZOH can be represented as a linear filter with a pulse-like impulse response

$$h(t) = \begin{cases} 1 & 0 \leq t < \Delta T \\ 0 & \text{otherwise} \end{cases}$$

Hint: Consider the sampled input data waveform $f^*(t)$ as a weighted impulse train at intervals $\Delta T$,

$$f^*(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - n\Delta T).$$

(b) Find and sketch the frequency response function $H(j\omega)$ for the ZOH data reconstructor.

(c) Compare $H(j\omega)$ with the frequency response of the ideal (cardinal) data reconstructor, and comment on why the ZOH is a non-ideal reconstructor.

(d) The ZOH reconstruction (see the fig above) seems to be slightly delayed from $f(t)$. Determine the delay.
Problem 6: (15 points)
The significant frequency range of an analog signal extends to 10 kHz. Beyond 10 kHz the signal spectrum rolls-off (attenuates) at a rate of 20 dB/decade.

The signal is to be sampled at a rate of 200 kHz. The aliased frequency components introduced into the 10 kHz range of interest must be kept below -60 dB as compared to signal components.

Suppose we use an analog low-pass pre-aliasing filter whose passband is flat over the 10 kHz band, and then attenuates at a rate steep enough to satisfy the above sampling requirements. What is this attenuation rate in dB/decade? What would be the minimum order of a low-pass filter to satisfy this condition?

Some useful/useless information:

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\[
\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b) \\
\sin(a - b) = \sin(a) \cos(b) - \cos(a) \sin(b) \\
\cos(a + b) = \cos(a) \cos(b) - \cos(a) \sin(b) \\
\cos(a - b) = \cos(a) \cos(b) + \cos(a) \sin(b) \\
\tan(a + b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a) \tan(b)} \\
\tan(a - b) = \frac{\tan(a) - \tan(b)}{1 + \tan(a) \tan(b)}
\]