Reading:

- Proakis & Manolakis, Chapter 3 (The z-transform)
- Oppenheim, Schafer & Buck, Chapter 3 (The z-transform)

1 The Discrete-Time Transfer Function

Consider the discrete-time LTI system, characterized by its pulse response \( \{h_n\} \):

\[
\begin{align*}
\{f_n\} & \xrightarrow{\text{convolution}} \{y_n\} = \{f_n \otimes h_n\} \\
F(z) & \hspace{1cm} H(z) \xrightarrow{\text{multiplication}} Y(z) = F(z)H(z)
\end{align*}
\]

We saw in Lec. 13 that the output to an input sequence \( \{f_n\} \) is given by the convolution sum:

\[
y_n = f_n \otimes h_n = \sum_{k=-\infty}^{\infty} f_k h_{n-k} = \sum_{k=-\infty}^{\infty} h_k f_{n-k},
\]

where \( \{h_n\} \) is the pulse response. Using the convolution property of the z-transform we have at the output

\[
Y(z) = F(z)H(z)
\]

where \( F(z) = \mathcal{Z}\{f_n\} \), and \( H(z) = \mathcal{Z}\{h_n\} \). Then

\[
H(z) = \frac{Y(z)}{F(z)}
\]

is the discrete-time transfer function, and serves the same role in the design and analysis of discrete-time systems as the Laplace based transfer function \( H(s) \) does in continuous systems.

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In general, for LTI systems the transfer function will be a rational function of \( z \), and may be written in terms of \( z \) or \( z^{-1} \), for example

\[
H(z) = \frac{N(s)}{D(s)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}
\]

where the \( b_i \), \( i = 0, \ldots, m \), \( a_i \), \( i = 0, \ldots, n \) are constant coefficients.

### 2 The Transfer Function and the Difference Equation

As defined above, let

\[
H(z) = \frac{Y(z)}{F(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}
\]

and rewrite as

\[
(a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}) Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}) F(z)
\]

If we apply the \( z \)-transform time-shift property

\[
Z \{f_{n-k}\} = z^{-k} F(z)
\]

term-by-term on both sides of the equation, (effectively taking the inverse \( z \)-transform)

\[
a_0 y_n + a_1 y_{n-1} + a_2 y_{n-2} + \ldots + a_N y_{n-N} = b_0 f_n + b_1 f_{n-1} + b_2 f_{n-2} + \ldots + b_M f_{n-M}
\]

and solve for \( y_n \)

\[
y_n = -\frac{1}{a_0} (a_1 y_{n-1} + a_2 y_{n-2} + \ldots + a_N y_{n-N}) + \frac{1}{a_0} (b_0 f_n + b_1 f_{n-1} + b_2 f_{n-2} + \ldots + b_M f_{n-M})
\]

\[
= \sum_{i=1}^{N} \left( -\frac{a_i}{a_0} \right) y_{n-i} + \sum_{i=0}^{M} \left( \frac{b_i}{a_0} \right) f_{n-i}
\]

which is in the form of a recursive linear difference equation as discussed in Lecture 13.

The transfer function \( H(z) \) directly defines the computational difference equation used to implement a LTI system.

#### Example 1

Find the difference equation to implement a causal LTI system with a transfer function

\[
H(z) = \frac{(1 - 2z^{-1})(1 - 4z^{-1})}{z(1 - \frac{1}{2}z^{-1})}
\]

**Solution:**

\[
H(z) = \frac{z^{-3} - 6z^{-2} + 8z^{-3}}{1 - \frac{1}{2}z^{-1}}
\]
from which
\[ y_n - \frac{1}{2} y_{n-1} = f_{n-1} - 6f_{n-2} + 8f_{n-3}, \]
or
\[ y_n = \frac{1}{2} y_{n-1} + (f_{n-1} - 6f_{n-2} + 8f_{n-3}). \]

The reverse holds as well: if we are given the difference equation, we can define the system transfer function.

### Example 2

Find the transfer function (expressed in powers of \( z \)) for the difference equation

\[ y_n = 0.25y_{n-2} + 3f_n - 3f_{n-1} \]

and plot the system poles and zeros on the \( z \)-plane.

**Solution:** Taking the \( z \)-transform of both sides

\[ Y(z) = 0.25z^{-2}Y(z) + 3F(z) - 3z^{-1}F(z) \]

and reorganizing

\[ H(z) = \frac{Y(z)}{F(z)} = \frac{3(1 - z^{-1})}{1 - 0.25z^{-2}} = \frac{3z(z - 1)}{z^2 - 0.25} \]

which has zeros at \( z = 0, 1 \) and poles at \( z = -0.5, 0.5 \):

![z-plane diagram](image-url)
3 Introduction to z-plane Stability Criteria

The stability of continuous time systems is governed by pole locations - for a system to be BIBO stable all poles must lie in the l.h. s-plane. Here we do a preliminary investigation of stability of discrete-time systems, based on z-plane pole locations of $H(z)$.

Consider the pulse response $h_n$ of the causal system with

$$H(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

with a single real pole at $z = a$ and with a difference equation

$$y_n = ay_{n-1} + f_n.$$ 

Clearly the pulse response is

$$h_n = \begin{cases} 
1 & n = 0 \\
 a^n & n \geq 1 
\end{cases}$$

The nature of the pulse response will depend on the pole location:

- $0 < a < 1$: In this case $h_n = a^n$ will be a decreasing function of $n$ and $\lim_{n \to \infty} h_n = 0$ and the system is **stable**.

- $a = 1$: The difference equation is $y_n = y_{n-1} + f_n$ (the system is a summer and the impulse response is $h_n = 1$, (non-decaying). The system is **marginally stable**.

- $a > 1$: In this case $h_n = a^n$ will be a increasing function of $n$ and $\lim_{n \to \infty} h_n = \infty$ and the system is **unstable**.

- $-1 < a < 0$: In this case $h_n = a^n$ will be a oscillating but decreasing function of $n$ and $\lim_{n \to \infty} h_n = 0$ and the system is **stable**.

- $a = -1$: The difference equation is $y_n = -y_{n-1} + f_n$ and the impulse response is $h_n = (-1)^n$, that is a pure oscillator. The system is **marginally stable**.

- $a < -1$: In this case $h_n = a^n$ will be a oscillating but increasing function of $n$ and $\lim_{n \to \infty} |h_n| = \infty$ and the system is **unstable**.
This simple demonstration shows that this system is stable only for the pole position \(-1 < a < 1\). In general for a system

\[ H(z) = K \prod_{k=1}^{M} \frac{z - z_k}{z - p_k} \]

having complex conjugate poles \((p_k)\) and zeros \((z_k)\):

A discrete-time system will be stable only if all of the poles of its transfer function \(H(z)\) lie within the unit circle on the \(z\)-plane.

4 The Frequency Response of Discrete-Time Systems

Consider the response of the system \(H(z)\) to an infinite complex exponential sequence

\[ f_n = A e^{j\omega n} = A \cos(\omega n) + jA \sin(\omega n), \]

where \(\omega\) is the normalized frequency (rad/sample). The response will be given by the convolution

\[ y_n = \sum_{k=-\infty}^{\infty} h_k f_{n-k} = \sum_{k=-\infty}^{\infty} h_k (A e^{j\omega(n-k)}) \]

\[ = A \left( \sum_{k=-\infty}^{\infty} h_k e^{-j\omega k} \right) e^{j\omega n} \]

\[ = AH(e^{j\omega})e^{j\omega n} \]

where the frequency response function \(H(e^{j\omega})\) is

\[ H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} \]

that is

The frequency response function of a LTI discrete-time system is \(H(z)\) evaluated on the unit circle - provided the ROC includes the unit circle. For a stable causal system this means there are no poles lying on the unit circle.
Alternatively, the frequency response may be based on a physical frequency $\Omega$ associated with an implied sampling interval $\Delta T$, and

$$H(e^{j\Omega\Delta T}) = H(z)|_{z=e^{j\Omega\Delta T}}$$

which is again evaluated on the unit circle, but at angle $\Omega\Delta T$.

From the definition of the DTFT based on a sampling interval $\Delta T$

$$H^*(j\Omega) = \sum_{n=0}^{\infty} h_n e^{-mj\Omega\Delta T} = H(z)|_{z=e^{-mj\Omega\Delta T}}$$

we can define the mapping between the imaginary axis in the $s$-plane and the unit-circle in the $z$-plane

$$s = j\Omega_o \longleftrightarrow z = e^{j\Omega_o\Delta T}$$

The periodicity in $H(e^{j\Omega\Delta T})$ can be clearly seen, with the “primary” strip in the $s$-plane (defined by $-\pi/\Delta T < \Omega < \pi/\Delta T$) mapping to the complete unit-circle. Within the primary strip, the l.h. $s$-plane maps to the interior of the unit circle in the $z$-plane, while the r.h. $s$-plane maps to the exterior of the unit-circle.
Aside: We use the argument to differentiate between the various classes of transfer functions:

<table>
<thead>
<tr>
<th></th>
<th>(H(s))</th>
<th>(H(j\Omega))</th>
<th>(H(z))</th>
<th>(H(e^{j\omega}))</th>
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<tbody>
<tr>
<td>Continuous Transfer Function</td>
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### 5 The Inverse \(z\)-Transform

The formal definition of the inverse \(z\)-transform is as a contour integral in the \(z\)-plane,

\[
\frac{1}{2\pi j} \oint_{-\infty}^{\infty} F(z) z^{n-1} \, dz
\]

where the path is a ccw contour enclosing all of the poles of \(F(z)\).

Cauchy’s residue theorem states

\[
\frac{1}{2\pi j} \oint_{-\infty}^{\infty} F(z) \, dz = \sum_k \text{Res} [F(z), p_k]
\]

where \(F(z)\) has \(N\) distinct poles \(p_k\), \(k = 1, \ldots, N\) and ccw path lies in the ROC.

For a simple pole at \(z = z_o\)

\[
\text{Res} [F(z), z_o] = \lim_{z \to z_o} (z - z_o) F(z),
\]

and for a pole of multiplicity \(m\) at \(z = z_o\)

\[
\text{Res} [F(z), z_o] = \lim_{z \to z_o} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z - z_o)^m F(z)
\]

The inverse \(z\)-transform of \(F(z)\) is therefore

\[
f_n = Z^{-1} \{F(z)\} = \sum_k \text{Res} [F(z)z^{n-1}, p_k].
\]

**Example 3**

A first-order low-pass filter is implemented with the difference equation

\[
y_n = 0.8y_{n-1} + 0.2f_n.
\]

Find the response of this filter to the unit-step sequence \(\{u_n\}\).
**Solution:** The filter has a transfer function

\[ H(z) = \frac{Y(z)}{F(z)} = \frac{0.2}{1 - 0.8z^{-1}} = \frac{0.2z}{z - 0.8} \]

The input \( \{f_n\} = \{u_n\} \) has a z-transform

\[ F(z) = \frac{z}{z - 1} \]

so that the z-transform of the output is

\[ Y(z) = H(z)U(z) = \frac{0.2z^2}{(z - 1)(z - 0.8)} \]

and from the Cauchy residue theorem

\[ y_n = \text{Res} \left[ Y(z)z^{n-1}, 1 \right] + \text{Res} \left[ Y(z)z^{n-1}, 0.8 \right] \]
\[ = \lim_{z \to 1}(z - 1)Y(z)z^{n-1} + \lim_{z \to 0.8}(z - 0.8)Y(z)z^{n-1} \]
\[ = \lim_{z \to 1} \frac{0.2z^{n+1}}{z - 0.8} + \lim_{z \to 0.8} \frac{0.2z^{n+1}}{z - 1} \]
\[ = 1 - 0.8^{n+1} \]

which is shown below

---

**Example 4**

Find the impulse response of the system with transfer function

\[ H(z) = \frac{1}{1 + z^{-2}} = \frac{z^2}{z^2 + 1} = \frac{z^2}{(z + j1)(z - j1)} \]
**Solution:** The system has a pair of imaginary poles at \( z = \pm j 1 \). From the Cauchy residue theorem

\[
    h_n = \mathcal{Z}^{-1} \{ H(z) \} = \text{Res} \left[ H(z)z^{n-1}, j1 \right] + \text{Res} \left[ H(z)z^{n-1}, -j1 \right]
\]

\[
    = \lim_{z \to j1} \frac{z^{n+1}}{z + j1} + \lim_{z \to -j1} \frac{z^{n+1}}{z - j1}
\]

\[
    = \frac{1}{j} \left( j1 \right)^{n+1} - \frac{1}{j} \left( -j1 \right)^{n+1}
\]

\[
    = \frac{j}{2} \left( 1 + (-1)^{n+1} \right)
\]

\[
    h_n = \begin{cases} 
        0 & n \text{ odd} \\
        (-1)^{n/2} & n \text{ even}
    \end{cases}
\]

where we note that the system is a pure oscillator (poles on the unit circle) with a frequency of half the Nyquist frequency.

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**Example 5**

Find the impulse response of the system with transfer function

\[
    H(z) = \frac{1}{1 + 2z + z^{-2}} = \frac{z^2}{z^2 + 2z + 1} = \frac{z^2}{(z + 1)^2}
\]

**Solution:** The system has a pair of coincident poles at \( z = -1 \). The residue at \( z = -1 \) must be computed using

\[
    \text{Res} \left[ F(z), z_o \right] = \lim_{z \to z_o} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z - z_o)^n F(z).
\]

With \( m = 2 \), at \( z = -1 \),

\[
    \text{Res} \left[ H(z)z^{n-1}, -1 \right] = \lim_{z \to -1} \frac{1}{(1)!} \frac{d}{dz} (z - 1)^2 H(z)z^{n-1}
\]

\[
    = \lim_{z \to -1} \frac{d}{dz} z^{n+1}
\]

\[
    = (n + 1)(-1)^n
\]

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The impulse response is

\[ h_n = \mathcal{Z}^{-1}\{H(z)\} = \text{Res}\left[H(z)z^{n-1}, -1\right] = (n + 1)(-1)^n. \]

Other methods of determining the inverse \( z \)-transform include:

**Partial Fraction Expansion:** This is a table look-up method, similar to the method used for the inverse Laplace transform. Let \( F(z) \) be written as a rational function of \( z^{-1} \):

\[
F(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{\prod_{k=1}^{M} (1 - c_i z^{-1})}{\prod_{k=1}^{N} (1 - d_i z^{-1})}
\]

If there are no repeated poles, \( F(z) \) may be expressed as a set of partial fractions.

\[
F(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}
\]

where the \( A_k \) are given by the residues at the poles

\[
A_k = \lim_{z \to d_k} (1 - d_k z^{-1})F(z).
\]

Since

\[
(d_k)^n u_n \xrightarrow{\mathcal{Z}} \frac{1}{1 - d_k z^{-1}}
\]

\[
f_n = \left(\sum_{k=1}^{N} A_k (d_k)^n\right) u_n.
\]
Example 6

Find the response of the low-pass filter in Ex. 3 to an input

\[ f_n = (-0.5)^n \]

Solution: From Ex. 3, and from the z-transform of \( \{f_n\} \),

\[ F(z) = \frac{1}{1 - 0.5z^{-1}}, \quad H(z) = \frac{0.2}{1 - 0.8z^{-1}} \]

so that

\[ Y(z) = \frac{0.2}{(1 + 0.5z^{-1})(1 - 0.8z^{-1})} \]

\[ = \frac{A_1}{1 + 0.5z^{-1}} + \frac{A_2}{1 - 0.8z^{-1}} \]

Using residues

\[ A_1 = \lim_{z \to -0.5} \frac{0.2}{1 - 0.8z^{-1}} = \frac{0.1}{1.3} \]

\[ A_2 = \lim_{z \to 0.8} \frac{0.2}{1 + 0.5z^{-1}} = \frac{0.16}{1.3} \]

and

\[ y_n = \frac{0.1}{1.3} z^{-1} \left\{ \frac{1}{1 + 0.5z^{-1}} \right\} + \frac{0.16}{1.3} z^{-1} \left\{ \frac{1}{1 - 0.8z^{-1}} \right\} \]

\[ = \frac{0.1}{1.3} (-0.5)^n + \frac{0.16}{1.3} (0.8)^n \]

Note: (1) If \( F(z) \) contains repeated poles, the partial fraction method must be extended as in the inverse Laplace transform.
(2) For complex conjugate poles – combine into second-order terms.

Power Series Expansion: Since

\[ F(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n} \]

if \( F(z) \) can be expressed as a power series in \( z^{-1} \), the coefficients must be \( f_n \).
Example 7
Find $Z^{-1}\{\log(1 + az^{-1})\}$.

Solution: $F(z)$ is recognized as having a power series expansion:

$$F(z) = \log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}a^n}{n}z^{-n} \quad \text{for } |a| < |z|$$

Because the ROC defines a causal sequence, the samples $f_n$ are

$$f_n = \begin{cases} 0 & \text{for } n \leq 0 \\ \frac{(-1)^{n+1}a^n}{n} & \text{for } n \geq 1. \end{cases}$$

Polynomial Long Division: For a causal system, with a transfer function written as a rational function, the first few terms in the sequence may sometimes be computed directly using polynomial division. If $F(z)$ is written as

$$F(z) = \frac{N(z^{-1})}{D(z^{-1})} = f_0 + f_1z^{-1} + f_2z^{-2} + f_2z^{-2} + \cdots$$

the quotient is a power series in $z^{-1}$ and the coefficients are the sample values.

Example 8
Determine the first few terms of $f_n$ for

$$F(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}}$$

using polynomial long division.

Solution:

\[
\begin{array}{c}
1 + 4z^{-1} + 7z^{-2} + \cdots \\
1 - 2z^{-1} + z^{-2}
\end{array}
\]

\[
\begin{array}{c}
1 + 2z^{-1} \\
\hline
1 - 2z^{-1} + z^{-2}
\end{array}
\]

\[
\begin{array}{c}
4z^{-1} - z^{-2} \\
\hline
4z^{-1} - 8z^{-2} + 4z^{-3}
\end{array}
\]

\[
\begin{array}{c}
7z^{-2} - 4z^{-3}
\end{array}
\]

so that

$$F(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}} = 1 + 4z^{-1} + 7z^{-2} + \cdots$$

and in this case the general term is

$$f_n = 3n + 1 \quad \text{for } n \geq 0.$$
In general, the computation can become tedious, and it may be difficult to recognize the general term from the first few terms in the sequence.