2.161 Signal Processing: Continuous and Discrete
Fall 2008

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The Design of IIR Filters (continued)

1.1 Design by the Matched z-Transform (Root Matching)

Given a prototype continuous filter $H_p(s)$,

$$H_p(s) = K \prod_{k=1}^{M} \frac{(s - z_k)}{(s - p_k)}$$

with zeros $z_k$, poles $p_k$, and gain $K$, the matched z-transform method approximates the ideal mapping

$$H_p(s) \rightarrow H(z)|_{z = e^{\ast T}}$$

by mapping the poles and zeros

$$H(z) = K' \prod_{k=1}^{M} \frac{(z - e^{z_k T})}{(z - e^{p_k T})}$$

where $K'$ must be determined from some empirical response comparison between the prototype and digital filters. Note that an implicit assumption is that all $s$-plane poles and zeros must lie in the primary strip in the $s$-plane (that is $|\Im(s)| < \pi/T$). Poles/zeros on the $s$-plane imaginary axis will map to the unit circle, and left-half $s$-plane poles and zeros will map to the interior of the unit circle, preserving stability.
The steps in the design procedure are:

1. Determine the poles and zeros of the prototype filter \( H_p(s) \).
2. Map the poles and zeros to the \( z \)-plane using \( z = e^{sT} \).
3. Form the \( z \)-plane transfer function with the transformed poles/zeros.
4. Determine the gain constant \( K' \) by matching gains at some frequency (for a low-pass filter this is normally the low frequency response).
5. Add poles or zeros at \( z = 0 \) to adjust the delay of the filter (while maintaining causality).

■ Example 1

Use the matched \( z \)-transform method to design a filter based on the prototype first-order low-pass filter

\[ H_p(s) = \frac{a}{s + a}. \]

**Solution:** The prototype has a single pole at \( s = -a \), and therefore the digital filter will have a pole at \( z = e^{-aT} \). The transfer function is

\[ H(z) = K' \frac{1}{z - e^{-aT}}. \]

To find \( K' \), compare the low frequency gains of the two filters:

\[
\begin{align*}
\lim_{\Omega \to 0} H_p(j\Omega) &= 1 \\
\lim_{\Omega \to 0} H(e^{j\Omega}) &= \frac{K'}{1 - e^{-aT}}.
\end{align*}
\]

therefore choose \( K' = 1 - e^{-aT} \). Then

\[ H(z) = \frac{1 - e^{-aT}}{z - e^{-aT}} = \frac{(1 - e^{-aT})z^{-1}}{1 - e^{-aT}z^{-1}} \]

and the difference equation is

\[ y_n = e^{-aT}y_{n-1} + (1 - e^{-aT})f_{n-1}. \]

Note that this is not a minimum delay filter, because it does not use \( f_n \). Therefore we can optionally add a zero at the origin, and take

\[
H(z) = \frac{(1 - e^{-aT})z}{z - e^{-aT}} = \frac{(1 - e^{-aT})}{1 - e^{-aT}z^{-1}}
\]

as the final filter design.
Example 2

Use the matched z-transform method to design a second-order band-pass filter based on the prototype filter

\[ H_p(s) = \frac{s}{s^2 + 0.2s + 1} \]

with a sampling interval \( T = 0.5 \) sec. Make frequency response plots to compare the prototype and digital filters.

Solution: The prototype filter as a zero at \( s = 0 \), and a complex conjugate pole pair at \( s = -0.1 \pm j0.995 \), so that

\[
H(z) = K' \frac{z-1}{(z - e^{(-0.1+j0.995)T})(z - e^{(-0.1-j0.995)T})}
\]

\[
= K' \frac{z-1}{z^2 - 1.6718z + 0.9048}
\]

To find \( K' \), compare the gains at \( \Omega = 1 \) rad/s (the peak response of \( H_p(j \Omega) \)),

\[
|H_p(j \Omega)|_{\Omega=1} = 5
\]

\[
|H(e^{j\Omega}T)|_{\Omega=1} = 10.54K'.
\]

and to match the gains \( K' = 5/10.54 = 0.4612 \), and

\[
H(z) = \frac{0.4612(z-1)}{z^2 - 1.6718z + 0.9048}
\]

![Frequency response plots](image-url)
To create a minimum delay filter, make the order of the numerator and denominator equal by adding a zero at the origin,

\[ H(z) = \frac{0.4612z(z - 1)}{z^2 - 1.6718z + 0.9048} = \frac{0.4612(1 - z^{-1})}{1 - 1.6718z^{-1} + 0.9048z^{-2}} \]

and implement the filter as

\[ y_n = 1.6718y_{n-1} - 0.9048y_{n-2} + 0.4612(f_n - f_{n-1}). \]

### 1.2 Design by the Bilinear Transform

As noted above, the ideal mapping of a prototype filter to the \( z \)-plane is

\[ H_p(s) \rightarrow H(z) \big|_{z=e^{sT}} \]

or

\[ s \rightarrow \frac{1}{T} \ln(z) \]

so that

\[ H(z) = H_p(s) \big|_{s=\frac{1}{T} \ln(z)} \]

The Laurent series expansion for \( \ln(z) \) is

\[ \ln(z) = 2 \left[ \frac{z - 1}{z + 1} + \frac{1}{3} \left( \frac{z - 1}{z + 1} \right)^3 + \frac{1}{5} \left( \frac{z - 1}{z + 1} \right)^5 + \cdots \right] \quad \text{for } \Re \{z\} \geq 0, z \neq 0. \]

The bilinear transform method uses the truncated series approximation

\[ s \rightarrow \frac{1}{T} \ln(z) \approx \frac{2}{T} \left( \frac{z - 1}{z + 1} \right) \]

In a more general sense, any transformation of the form

\[ s = A \left( \frac{z - 1}{z + 1} \right) \]

which implies

\[ z = - \left( \frac{s + A}{s - A} \right) \]

is a bilinear transform. In particular, when \( A = 2/T \) the method is known as Tustin’s method.

With this transformation the digital filter is designed from the prototype using

\[ H(z) = H_p(s) \big|_{s=\frac{2}{T} \left( \frac{z - 1}{z + 1} \right)} \]
Example 3

Find the bilinear transform equivalent of an integrator

\[ H_p(s) = \frac{1}{s}. \]

Solution:

\[ H(z) = \left( \frac{1}{s} \right) \bigg|_{s = \frac{T}{2} \left( \frac{z - 1}{1 - z^{-1}} \right)} = \left( \frac{T}{2} \right) \frac{1 + z^{-1}}{1 - z^{-1}} \]

and the difference equation is

\[ y_n = y_{n-1} + \frac{T}{2} (f_n + f_{n-1}) \]

which is the classical trapezoidal (or mid-point) rule for numerical integration.

The bilinear transform maps the left half s-plane to the interior of the unit circle, and thus preserves stability. In addition, we will see below that it maps the entire imaginary axis of the s-plane to the unit circle, and thus avoids aliasing in the frequency response.

Thus every point on the frequency response of the continuous-time prototype filter, is mapped to a corresponding point in the frequency response of the discrete-time filter, although with a different frequency. This means that every feature in the frequency response of the prototype filter is preserved, with identical gain and phase shift, at some frequency the digital filter.

Example 4

Find the bilinear transform equivalent of a first-order low-pass filter

\[ H_p(s) = \frac{a}{s + a}. \]
Solution:

\[ H(z) = \left( \frac{a}{s+a} \right) \bigg|_{s=\frac{2}{T} (\frac{z-1}{z+1})} = \frac{(aT/2)(z+1)}{(z-1)+(aT/2)(z+1)} = \frac{(aT/2)(1+z^{-1})}{(1+aT/2) - (1-aT/2)z^{-1}} \]

and the difference equation is

\[ y_n = \frac{1-aT/2}{1+aT/2} y_{n-1} + \frac{aT/2}{1+aT/2} f_n. \]

Comparing the frequency responses of the two filters,

\[ H(e^{j\Omega T})\big|_{\Omega=0} = 1 \neq 0 = H_p(j 0) \]

\[ \lim_{\Omega \to \pi/T} H(e^{j\Omega T}) = 0 \neq 0 = \lim_{\Omega \to \infty} H_p(j \Omega), \]

demonstrating the assertion above that the entire frequency response of the prototype filter has been transformed to the unit circle.

**1.2.1 Frequency Warping in the Bilinear Transform**

The mapping

\[ s \leftrightarrow \frac{2}{T} \left( \frac{z-1}{z+1} \right) \]

implies that when \( z = e^{j\Omega T}, \)

\[ s = \frac{2}{T} \left( \frac{e^{j\Omega T} - 1}{e^{j\Omega T} + 1} \right) = \frac{2}{T} \tan \left( \frac{\Omega T}{2} \right) \]

so that

\[ H(e^{j\Omega T}) = H_p \left( \frac{2}{T} \tan \left( \frac{\Omega T}{2} \right) \right) \]

which gives a nonlinear warping of the frequency scales in the frequency response of the two filters.
In particular

\[ H(e^{j0}) = H_p(j0), \quad \text{and} \quad H(e^{j\pi}) = H_p(j\infty) \]

and there is no aliasing in the frequency response.

### 1.2.2 Pre-warping of Critical Frequencies in Bilinear Transform Filter Design

The specifications for a digital filter must be done in the digital domain, that is the critical band-edge frequencies must relate to the performance of the final design - not the continuous prototype.

Therefore, in designing the continuous prototype we need to choose band-edge frequencies that will warp to the correct values after the bilinear transform. This procedure is known as \textit{pre-warping}. For example, if we are given a specification for a digital low-pass filter such as
we would pre-warp the frequencies $\Omega_c$ and $\Omega_r$ to

$$\Omega'_c = \frac{2}{T} \tan \frac{\Omega_c T}{2}, \quad \text{and} \quad \Omega'_r = \frac{2}{T} \tan \frac{\Omega_r T}{2}$$

and design the prototype to meet the specifications with $\Omega'_c$ and $\Omega'_r$ as the band edges.

**Design Procedure:** For any class of filter (band-pass, band-stop) the procedure is:

1. Define all band-edge critical frequencies for the digital filter.
2. Pre-warp all critical frequencies using $\Omega' = (T/2) \tan(\Omega T/2)$.
3. Design the continuous prototype using the pre-warped frequencies.
4. Use the bilinear transform to transform $H_p(s)$ to $H(z)$.
5. Realize the digital filter as a difference equation.

**Example 5**

Use the bilinear transform method to design a low-pass filter, with $T = .01$ sec., based on a prototype Butterworth filter to meet the following specifications.

![Diagram showing frequency response](image)

**Solution:** Pre-warp the band-edges:

$$\Omega'_c = \frac{2}{T} \tan \left( \frac{\Omega_c T}{2} \right) = 64.9839 \text{ rad/s}$$

$$\Omega'_r = \frac{2}{T} \tan \left( \frac{\Omega_r T}{2} \right) = 145.3085 \text{ rad/s}.$$  

From the specifications $\epsilon = 0.3333$ and $\lambda = 4.358$, and the required order for the prototype Butterworth filter is

$$N \geq \frac{\log(\lambda/\epsilon)}{\log(\Omega'_r/\Omega'_c)} = 3.1946$$

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so take \( N = 4 \). The four poles \((p_1, \ldots, p_4)\) lie on a circle of radius \( \Omega_c e^{-1/N} = 82.526\),

\[
|p_n| = 82.526, \\
\angle p_n = \pi (2n + 3)/8
\]

for \( n = 1 \ldots 4 \). The prototype transfer function is

\[
H_p(s) = \frac{p_1 p_2 p_3 p_4}{(s - p_1)(s - p_2)(s - p_3)(s - p_4)} = \frac{5.3504 \times 10^7}{s^4 + 223.4897 s^3 + 24974 s^2 + 1.6348 \times 10^6 s + 5.3504 \times 10^7}.
\]

Applying the bilinear transform

\[
H(z) = H_p(s) \big|_{s = \frac{z - 1}{T(z + 1)}}
\]

gives

\[
H(z) = \frac{0.0112 (1 + z^{-1})^4}{1.0000 - 1.9105 z^{-1} + 1.6620 z^{-2} - 0.6847 z^{-3} + 0.1128 z^{-4}}
\]

and the frequency response of the digital filter (as a power gain) is shown below: