2.161 Signal Processing: Continuous and Discrete
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

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Lecture 19[1]

Reading:

- Proakis and Manolakis: Sec. 10.3.3
- Oppenheim, Schafer, and Buck: Sec. 7.1

1 The Design of IIR Filters (continued)

1.1 Design by the Matched z-Transform (Root Matching)

Given a prototype continuous filter $H_p(s)$,

$$H_p(s) = K\frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - p_k)}$$

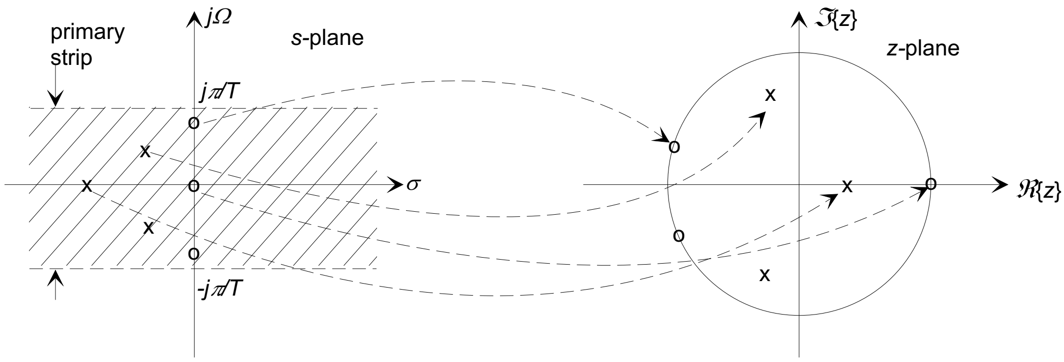
with zeros z_k, poles p_k, and gain K, the matched z-transform method approximates the ideal mapping

$$H_p(s) \longrightarrow H(z)|_{z=\mathrm{e}^{sT}}$$

by mapping the poles and zeros

$$H(z) = K'\frac{\prod_{k=1}^{M}(z - \mathrm{e}^{z_kT})}{\prod_{k=1}^{N}(z - \mathrm{e}^{p_kT})}$$

where K' must be determined from some empirical response comparison between the prototype and digital filters. Note that an implicit assumption is that all s-plane poles and zeros must lie in the primary strip in the s-plane (that is $|\Im(s)| < \pi/T$). Poles/zeros on the s-plane imaginary axis will map to the unit circle, and left-half s-plane poles and zeros will map to the interior of the unit circle, preserving stability.



The steps in the design procedure are:

1. Determine the poles and zeros of the prototype filter $H_p(s)$.
2. Map the poles and zeros to the z-plane using $z = \mathrm{e}^{sT}$.
3. Form the z-plane transfer function with the transformed poles/zeros.
4. Determine the gain constant K' by matching gains at some frequency (for a low-pass filter this is normally the low frequency response).
5. Add poles or zeros at $z = 0$ to adjust the delay of the filter (while maintaining causality).

■ Example 1

Use the matched z-transform method to design a filter based on the prototype first-order low-pass filter

$$H_p(s) = \frac{a}{s+a}.$$

Solution: The prototype has a single pole at $s = -a$, and therefore the digital filter will have a pole at $z = \mathrm{e}^{-aT}$. The transfer function is

$$H(z) = K' \frac{1}{z - \mathrm{e}^{-aT}}.$$

To find K', compare the low frequency gains of the two filters:

$$\begin{aligned} \lim_{\Omega \to 0} H_p(\mathrm{j}\,\Omega) &= 1 \\ \lim_{\Omega \to 0} H(\mathrm{e}^{\mathrm{j}\,\Omega}) &= \frac{K'}{1 - \mathrm{e}^{-aT}}, \end{aligned}$$

therefore choose $K' = 1 - \mathrm{e}^{-aT}$. Then

$$H(z) = \frac{1 - \mathrm{e}^{-aT}}{z - e^{-aT}} = \frac{(1 - \mathrm{e}^{-aT})z^{-1}}{1 - e^{-aT}z^{-1}}$$

and the difference equation is

$$y_n = \mathrm{e}^{-aT} y_{n-1} + (1 - \mathrm{e}^{-aT}) f_{n-1}.$$

Note that this is not a minimum delay filter, because it does not use f_n. Therefore we can optionally add a zero at the origin, and take

$$\boxed{H(z) = \frac{(1 - \mathrm{e}^{-aT})z}{z - e^{-aT}} = \frac{(1 - \mathrm{e}^{-aT})}{1 - e^{-aT}z^{-1}}}$$

as the final filter design.

■ Example 2

Use the matched z-transform method to design a second-order band-pass filter based on the prototype filter

$$H_p(s) = \frac{s}{s^2 + 0.2s + 1}$$

with a sampling interval $T = 0.5$ sec. Make frequency response plots to compare the prototype and digital filters.

Solution: The prototype filter as a zero at $s = 0$, and a complex conjugate pole pair at $s = -0.1 \pm \mathrm{j}\,0.995$, so that

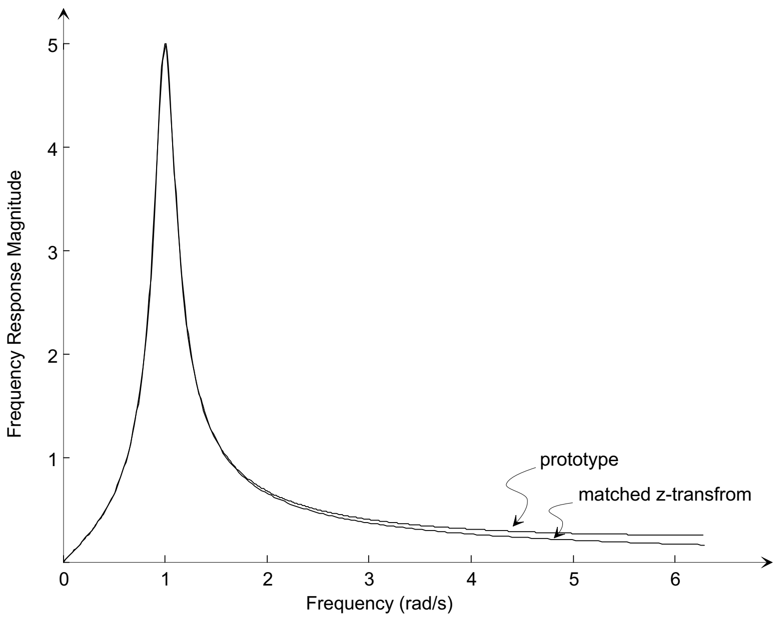
$$\begin{aligned} H(z) &= K' \frac{z-1}{(z - \mathrm{e}^{(-0.1+\mathrm{j}\,0.995)T})(z - \mathrm{e}^{(-0.1-\mathrm{j}\,0.995)T})} \\ &= K' \frac{z-1}{z^2 - 1.6718z + 0.9048} \end{aligned}$$

To find K', compare the gains at $\Omega = 1$ rad/s (the peak response of $H_p(\mathrm{j}\,\Omega)$),

$$\begin{aligned} \left|H_p(\mathrm{j}\,\Omega)\right|_{\Omega=1} &= 5 \\ \left|H(\mathrm{e}^{\mathrm{j}\,\Omega T})\right|_{\Omega=1} &= 10.54K'. \end{aligned}$$

and to match the gains $K' = 5/10.54 = 0.4612$, and

$$H(z) = \frac{0.4612(z-1)}{z^2 - 1.6718z + 0.9048}$$



To create a minimum delay filter, make the order of the numerator and denominator equal by adding a zero at the origin,

$$H(z) = \frac{0.4612z(z-1)}{z^2 - 1.6718z + 0.9048} = \frac{0.4612(1-z^{-1})}{1 - 1.6718z^{-1} + 0.9048z^{-2}}$$

and implement the filter as

$$y_n = 1.6718y_{n-1} - 0.9048y_{n-2} + 0.4612(f_n - f_{n-1}).$$

1.2 Design by the Bilinear Transform

As noted above, the ideal mapping of a prototype filter to the z-plane is

$$H_p(s) \longrightarrow H(z)|_{z=\mathrm{e}^{sT}}$$

or

$$s \longrightarrow \frac{1}{T}\ln(z)$$

so that

$$H(z) = H_p(s)|_{s=\frac{1}{T}\ln(z)}.$$

The Laurent series expansion for $\ln(z)$ is

$$\ln(z) = 2\left[\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \cdots\right] \qquad \text{for } \Re\{z\} \geq 0, z \neq 0.$$

The bilinear transform method uses the truncated series approximation

$$s \longrightarrow \frac{1}{T}\ln(z) \approx \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

In a more general sense, any transformation of the form

$$s = A\left(\frac{z-1}{z+1}\right) \quad \text{which implies} \quad z = -\left(\frac{s+A}{s-A}\right)$$

is a bilinear transform. In particular, when $A = 2/T$ the method is known as Tustin's method.

With this transformation the digital filter is designed from the prototype using

$$H(z) = H_p(s)|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

■ Example 3

Find the bilinear transform equivalent of an integrator

$$H_p(s) = \frac{1}{s}.$$

Solution:

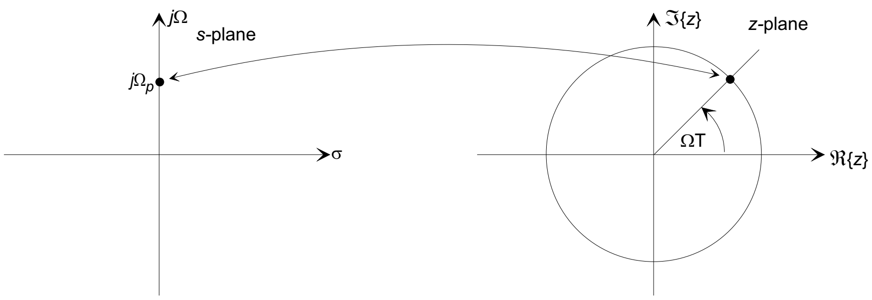
$$H(z) = \left(\frac{1}{s}\right)\Bigg|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)} = \left(\frac{T}{2}\right)\frac{1+z^{-1}}{1-z^{-1}}$$

and the difference equation is

$$y_n = y_{n-1} + \frac{T}{2}\left(f_n + f_{n-1}\right)$$

which is the classical trapezoidal (or mid-point) rule for numerical integration.

The bilinear transform maps the left half s-plane to the interior of the unit circle, and thus preserves stability. In addition, we will see below that it maps the *entire* imaginary axis of the s-plane to the unit circle, and thus avoids aliasing in the frequency response.



Thus every point on the frequency response of the continuous-time prototype filter, is mapped to a corresponding point in the frequency response of the discrete-time filter, although with a different frequency. This means that every feature in the frequency response of the prototype filter is preserved, with identical gain and phase shift, at some frequency the digital filter.

■ Example 4

Find the bilinear transform equivalent of a first-order low-pass filter

$$H_p(s) = \frac{a}{s+a}.$$

Solution:

$$\begin{aligned}
H(z) &= \left.\left(\frac{a}{s+a}\right)\right|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)} \\
&= \frac{(aT/2)(z+1)}{(z-1)+(aT/2)(z+1)} \\
&= \frac{(aT/2)(1+z^{-1})}{(1+aT/2)-(1-aT/2)z^{-1}}
\end{aligned}$$

and the difference equation is

$$y_n = \frac{1-aT/2}{1+aT/2} y_{n-1} + \frac{aT/2}{1+aT/2} f_n.$$

Comparing the frequency responses of the two filters,

$$H(\mathrm{e}^{\mathrm{j}\,\Omega T})\big|_{\Omega=0} = 1\angle 0 = H_p(\mathrm{j}\,0)$$

$$\lim_{\Omega\to\pi/T} H(\mathrm{e}^{\mathrm{j}\,\Omega T}) = 0\angle\left(-\frac{\pi}{2}\right) = \lim_{\Omega\to\infty} H_p(\mathrm{j}\,\Omega),$$

demonstrating the assertion above that the entire frequency response of the prototype filter has been transformed to the unit circle.

1.2.1 Frequency Warping in the Bilinear Transform

The mapping

$$s \longleftrightarrow \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

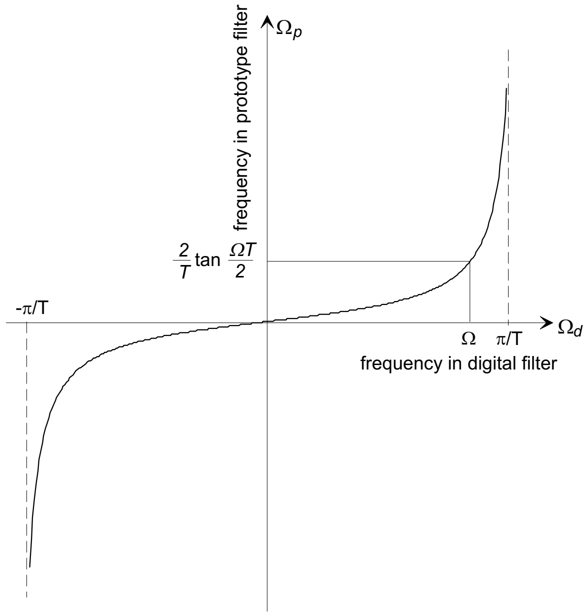
implies that when $z = \mathrm{e}^{\mathrm{j}\,\Omega T}$,

$$s = \frac{2}{T}\left(\frac{\mathrm{e}^{\mathrm{j}\,\Omega T}-1}{\mathrm{e}^{\mathrm{j}\,\Omega T}+1}\right) = \mathrm{j}\,\frac{2}{T}\tan\left(\frac{\Omega T}{2}\right)$$

so that

$$H(\mathrm{e}^{\mathrm{j}\,\Omega T}) = H_p\left(\mathrm{j}\,\frac{2}{T}\tan\left(\frac{\Omega T}{2}\right)\right)$$

which gives a nonlinear warping of the frequency scales in the frequency response of the two filters.



In particular

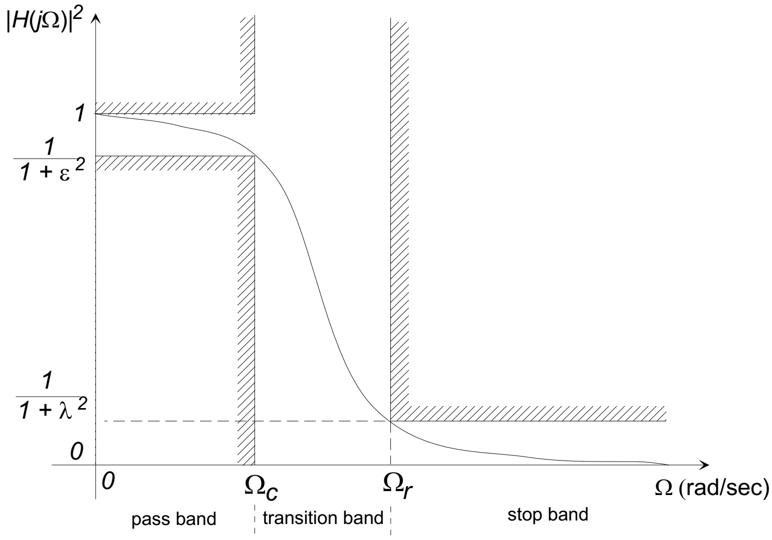
$$H(\mathrm{e}^{\mathrm{j}\,0}) = H_p(\mathrm{j}\,0)\,, \text{ and } H(\mathrm{e}^{\mathrm{j}\,\pi}) = H_p(\mathrm{j}\,\infty)$$

and there is no aliasing in the frequency response.

1.2.2 Pre-warping of Critical Frequencies in Bilinear Transform Filter Design

The specifications for a digital filter must be done in the digital domain, that is the critical band-edge frequencies must relate to the performance of the final design - not the continuous prototype.

Therefore, in designing the continuous prototype we need to choose band-edge frequencies that will warp to the correct values after the bilinear transform. This procedure is known as *pre-warping.* For example, if we are given a specification for a digital low-pass filter such as



we would pre-warp the frequencies Ω_c and Ω_r to

$$\Omega'_c = \frac{2}{T}\tan\frac{\Omega_c T}{2}, \quad \text{and} \quad \Omega'_r = \frac{2}{T}\tan\frac{\Omega_r T}{2}$$

and design the prototype to meet the specifications with Ω'_c and Ω'_c as the band edges.

Design Procedure: For any class of filter (band-pass, band-stop) the procedure is:

(1) Define all band-edge critical frequencies for the digital filter.

(2) Pre-warp all critical frequencies using $\Omega' = (T/2)\tan(\Omega T/2)$.

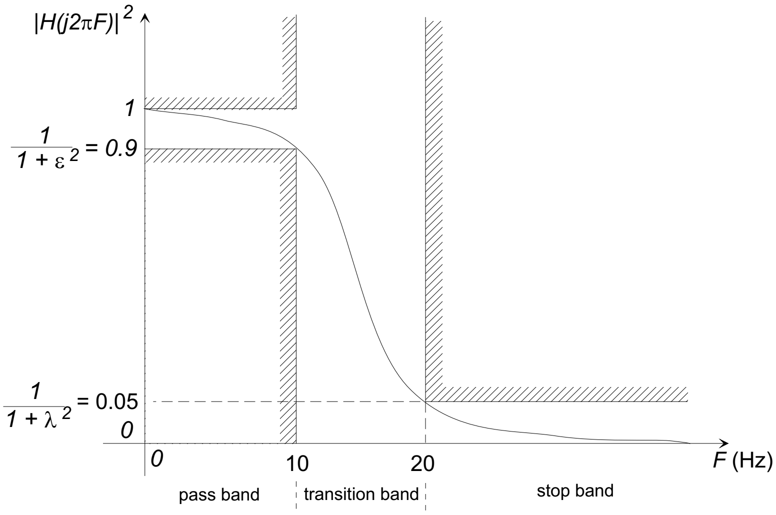
(3) Design the continuous prototype using the pre-warped frequencies.

(4) Use the bilinear transform to transform $H_p(s)$ to $H(z)$.

(5) Realize the digital filter as a difference equation.

■ Example 5

Use the bilinear transform method to design a low-pass filter, with $T = .01$ sec., based on a prototype Butterworth filter to meet the following specifications.



Solution: Pre-warp the band-edges:

$$\Omega'_c = \frac{2}{T}\tan\left(\frac{\Omega_c T}{2}\right) = 64.9839 \text{ rad/s}$$

$$\Omega'_r = \frac{2}{T}\tan\left(\frac{\Omega_r T}{2}\right) = 145.3085 \text{ rad/s.}$$

From the specifications $\epsilon = 0.3333$ and $\lambda = 4.358$, and the required order for the prototype Butterworth filter is

$$N \geq \frac{\log(\lambda/\epsilon)}{\log(\Omega'_r/\Omega'_c)} = 3.1946$$

so take $N = 4$. The four poles $(p_1, \ldots, p_4)$ lie on a circle of radius $\Omega_c' \epsilon^{-1/N} = 82.526$,

$$\begin{aligned} |p_n| &= 82.526, \\ \angle p_n &= \pi(2n+3)/8 \end{aligned}$$

for $n = 1 \ldots 4$. The prototype transfer function is

$$\begin{aligned} H_p(s) &= \frac{p_1 p_2 p_3 p_4}{(s-p_1)(s-p_2)(s-p_3)(s-p_4)} \\ &= \frac{5.3504 \times 10^7}{s^4 + 223.4897 s^3 + 24974 s^2 + 1.6348 \times 10^6 s + 5.3504 \times 10^7}. \end{aligned}$$

Applying the bilinear transform

$$H(z) = H_p(s)\big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

gives

$$H(z) = \frac{0.0112(1+z^{-1})^4}{1.0000 - 1.9105 z^{-1} + 1.6620 z^{-2} - 0.6847 z^{-3} + 0.1128 z^{-4}}$$

and the frequency response of the digital filter (as a power gain) is shown below: