3.18 Unsteady Motion - Added Mass

D’Alembert: ideal, irrotational, unbounded, steady.

Example Force on a sphere accelerating \( U = U(t) \), unsteady) in an unbounded fluid that is at rest at infinity.

K.B.C on sphere: \( \frac{\partial \phi}{\partial r} \bigg|_{r=a} = U(t) \cos \theta \)

Solution: Simply a 3D dipole (no stream)

\[
\phi = -U(t) \frac{a^3}{2r^2} \cos \theta
\]

Check: \( \frac{\partial \phi}{\partial r} \bigg|_{r=a} = U(t) \cos \theta \)
Hydrodynamic force:

\[ F_x = -\rho \int \int_B \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) n_x dS \]

On \( r = a \),

\[ \frac{\partial \phi}{\partial t} \bigg|_{r=a} = -\dot{U} \frac{a^3}{2r^2} \cos \theta \bigg|_{r=a} = -\frac{1}{2} \dot{U} a \cos \theta \]

\[ \nabla \phi \bigg|_{r=a} = \left( \frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \phi} \right) = \left( U \cos \theta, \frac{1}{2} U \sin \theta, 0 \right) \]

\[ |\nabla \phi|^2 \bigg|_{r=a} = U^2 \cos^2 \theta + \frac{1}{4} U^2 \sin^2 \theta; \hat{n} = -\hat{e}_r, n_x = -\cos \theta \]

\[ \int \int_B dS = \int_0^\pi (a \, d\theta) (2\pi a \sin \theta) \]
Finally,

\[
F_x = (-\rho) 2\pi a^2 \int_0^\pi d\theta \sin \theta \left( -\cos \theta \right) \left[ -\frac{1}{2} \dot{U} a \cos \theta + \frac{1}{2} \left( U^2 \cos^2 \theta + \frac{1}{4} U^2 \sin^2 \theta \right) \right] \\
F_x = -\dot{U} (\rho a^3) \pi \int_0^{\pi} d\theta \sin \theta \cos^2 \theta + (\rho U^2) \pi a^2 \int_0^{\pi} d\theta \sin \theta \cos \theta \left( \cos^3 \theta + \frac{1}{4} \sin^2 \theta \right)
\]

\[
\begin{array}{c}
F_x = -\dot{U}(t) \left[ \rho \frac{2\pi a^3}{3} \right]
\end{array}
\]

Thus the **Hydrodynamic Force** on a sphere of diameter \( a \) moving with velocity \( U(t) \) in an unbounded fluid of density \( \rho \) is given by

\[
F_x = -\dot{U}(t) \left[ \rho \frac{2\pi a^3}{3} \right]
\]

Comments:

- If \( \dot{U} = 0 \) \( \rightarrow F_x = 0 \), i.e., steady translation \( \rightarrow \) no force (D’Alembert’s Condition ok).
- \( F_x \propto \dot{U} \) with a (−) sign, i.e., the fluid tends to ‘resist’ the acceleration.
- \([\cdots]\) has the units of (fluid) mass \( \equiv m_a \)
- Equation of Motion for a body of mass \( M \) that moves with velocity \( U \):

\[
\begin{array}{c}
M \ddot{U} = \Sigma F = F_H + F_B \\
\text{Body mass} \quad \text{Hydrodynamic force} \quad \text{All other forces on body} \quad \text{Fluid mass}
\end{array}
\]

\[
(M + m_a) \ddot{U} = F_B
\]

i.e., the presence of fluid around the body acts as an **added** or **virtual** mass to the body.
3.19 General 6 Degrees of Freedom Motions

3.19.1 Notation Review

(3D) $U_1, U_2, U_3$: Translational velocities

$U_4 \equiv \Omega_1, U_5 \equiv \Omega_2, U_6 \equiv \Omega_3$: Rotational velocities

(2D) $U_1, U_2$: Translational velocities

$U_6 \equiv \Omega_3$: Rotational velocity

$U_3 = U_4 = U_5 = 0$

3.19.2 Added Mass Tensor (matrix)

$m_{ij}; i, j = 1, 2, 3, 4, 5, 6$

$m_{ij}$: associated with force on body in $i$ direction due to unit acceleration in $j$ direction. For example, for a sphere:

$m_{11} = m_{22} = m_{33} = \frac{1}{2} \rho \mathcal{V} = (m_A)$ all other $m_{ij} = 0$
3.19.3 Added Masses of Simple 2D Geometries

• Circle

\[ m_{11} = m_{22} = \rho \forall = \rho \pi a^2 \]

• Ellipse

\[ m_{11} = \rho \pi a^2, m_{22} = \rho \pi b^2 \]

• Plate

\[ m_{11} = \rho \pi a^2, m_{22} = 0 \]
- Square

\[ m_{11} = m_{22} \approx 4.754 \rho a^2 \]

A reasonable approximation to estimate the added mass of a 2D body is to use the displaced mass (\( \rho A \)) of an ‘equivalent cylinder’ of the same lateral dimension or one that ‘rounds off’ the body. For example, consider a square and approximate with an

(a) inscribed circle: \( m_A = \rho \pi a^2 = 3.14 \rho a^2 \).

(b) circumscribed circle: \( m_A = \rho \pi (\sqrt{2}a)^2 = 6.28 \rho a^2 \).

Arithmetic mean of (a) + (b) \( \approx 4.71 \rho a^2 \).
3.19.4 Generalized Forces and Moments

In this paragraph we are looking at the most general case where forces and moments are induced on rigid body moving with 6 DoF motions, in an unbounded fluid that is at rest at infinity.

**Body fixed reference frame**, i.e., $OX_1X_2X_3$ is fixed on the body.

Consider a body with a 6 DoF motion $(\vec{U}, \vec{\Omega})$, and a fixed reference frame $OX_1X_2X_3$. Then the hydrodynamic forces and moments with respect to $O$ are given by the following relations (JNN §4.13)

- **Forces**

  
  $$F_j = -\dot{U}_i m_{ji} - E_{jkl} U_k \Omega_l m_{li} \quad \text{with} \quad i = 1, 2, 3, 4, 5, 6$$

  and $j, k, l = 1, 2, 3$

- **Moments**

  $$M_j = -\dot{U}_i m_{j+3,i} - E_{jkl} U_k \Omega_l m_{i+3,l} - E_{jkl} U_k \Omega_l m_{li}$$

  \quad \text{with} \quad i = 1, 2, 3, 4, 5, 6

  \quad \text{and} \quad j, k, l = 1, 2, 3
Einstein’s $\Sigma$ notation applies.

\[ E_{ijkl} = \text{‘alternating tensor’} = \begin{cases} 
0 & \text{if any } j, k, l \text{ are equal} \\
1 & \text{if } j, k, l \text{ are in cyclic order, i.e.,} \\
(1, 2, 3), (2, 3, 1), \text{ or } (3, 1, 2) \\
-1 & \text{if } j, k, l \text{ are not in cyclic order i.e.,} \\
(1, 3, 2), (2, 1, 3), (3, 2, 1) 
\end{cases} \]

Note:

(a) if $\Omega_k \equiv 0$, $F_j = -\dot{U}_i m_{ji}$ (as expected by definition of $m_{ij}$).

Also if $\dot{U}_i \equiv 0$, then $F_j = 0$ for any $U_i$, no force in steady translation.

(b) $B_l \sim U_i m_{li}$ ‘added momentum’ due to rotation of axes.

Then all the terms marked as 2. are proportional to $\sim \Omega \times \vec{B}$ where $\vec{B}$ is linear momentum (momentum from $i$ coordinate into new $x_j$ direction).

(c) If $\Omega_k \equiv 0$: $M_j = -\dot{U}_i m_{j+3,i} m_{ij} - \underbrace{E_{jkl} U_k U_i m_{li}}_{\text{even with } \dot{U} = 0, \text{ } M_j \neq 0 \text{ due to this term}}$.

**Moment** on a body due to pure steady translation – ‘Munk’ moment.
3.19.5 Example Generalized motions, forces and moments.

A certain body has non-zero added mass coefficients only on the diagonal, i.e. $m_{ij} = \delta_{ij}$. For a body motion given by $U_1 = t$, $U_2 = -t$, and all other $U_i$, $\Omega_i = 0$, the forces and moments on the body in terms of $m_i$ are:

$F_1 = \ldots, F_2 = \ldots, F_3 = \ldots, M_1 = \ldots, M_2 = \ldots, M_3 = \ldots$

Solution:

$m_{ij} = \delta_{ij}$

$U_1 = t \quad U_2 = -t \quad U_i = 0 \quad i = 3, 4, 5, 6 \quad \Omega_k = 0 \quad k = 1, 2, 3$

$\dot{U}_1 = 1 \quad \dot{U}_2 = -1 \quad \dot{U}_i = 0 \quad i = 3, 4, 5, 6$

Use the relations from (JNN §4.13):

$F_j = -\dot{U}_i m_{ij} - E_{jkl} U_i \Omega_k m_{kl} \xrightarrow{\Omega_k = 0} \dot{F}_j = -\dot{U}_i m_{ij}$

$M_j = -\dot{U}_i m_{i(j+3)} - E_{jkl} U_i \Omega_k m_{i(l+3)} - E_{jkl} U_k U_i m_{li} \xrightarrow{\Omega_k = 0} \dot{M}_j = -\dot{U}_i m_{i(j+3)} - E_{jkl} U_k U_i m_{li}$

where $i = 1, 2, 3, 4, 5, 6$ and $j, k, l = 1, 2, 3$

For $F_1, F_2, F_3$ use the previous relationship for $F_j$ with $j = 1, 2, 3$ respectively:

$F_1 = -\dot{U}_1 m_{11} - \dot{U}_2 m_{21} - \dot{U}_3 m_{31} - \dot{U}_4 m_{41} - \dot{U}_5 m_{51} - \dot{U}_6 m_{61} \rightarrow F_1 = -m_{11}$

$F_2 \xrightarrow{\text{Check}} -\dot{U}_2 m_{22} \rightarrow F_2 = m_{22}$

$F_3 \xrightarrow{\text{Check}} -\dot{U}_3 m_{33} \rightarrow F_3 = 0$
For $M_1, M_2, M_3$ use the previous relationship for $M_j$ with $j = 1, 2, 3$ respectively:

\[
M_1 = -\dot{U}_i m_i(1+3) - E_{1kl} U_k U_i m_{li} \\
= -\dot{U}_i m_{i4} - E_{1kl} U_k U_i m_{li} \\
= -\dot{U}_1 m_{14} - \dot{U}_2 m_{24} - \dot{U}_3 m_{34} - \dot{U}_4 m_{44} - \dot{U}_5 m_{54} - \dot{U}_6 m_{64} \\
- E_{123} U_2 (U_1 m_{13} + U_2 m_{23} + U_3 m_{33} + U_4 m_{43} + U_5 m_{53} + U_6 m_{63}) \\
- E_{132} U_3 (U_1 m_{12} + U_2 m_{22} + U_3 m_{32} + U_4 m_{42} + U_5 m_{52} + U_6 m_{62}) \rightarrow M_1 = 0
\]

\[
M_2 = -\dot{U}_i m_{i5} - E_{2kl} U_k U_i m_{li} \\
= \dot{U}_5 m_{55} - E_{231} U_3 U_i m_{1i} - E_{213} U_1 U_i m_{3i} \\
= -E_{213} U_1 U_3 m_{33} \rightarrow M_2 = 0
\]

\[
M_3 = -\dot{U}_i m_{i6} - E_{3kl} U_k U_i m_{li} \\
= \dot{U}_6 m_{66} - E_{312} U_1 U_i m_{2i} - E_{321} U_2 U_i m_{1i} \\
= -U_1 U_2 m_{22} + U_2 U_1 m_{11} \rightarrow M_3 = t^2 (m_{22} - m_{11})
\]
3.19.6 Example Munk Moment on a 2D submarine in steady translation

\[ U_1 = U \cos \theta \]
\[ U_2 = -U \sin \theta \]

Consider steady translation motion: \( \dot{U} = 0; \Omega_k = 0 \). Then

\[ M_3 = -E_{3kl}U_k U_i m_{li} \]

For a 2D body, \( m_{3i} = m_{i3} = 0 \), also \( U_3 = 0, i, k, l = 1, 2 \). This implies that:

\[
M_3 = -\sum_{i=1}^{1} E_{312} U_1 (U_1 m_{21} + U_2 m_{22}) - E_{321} U_2 (U_1 m_{11} + U_2 m_{12}) = -U_1 U_2 (m_{22} - m_{11}) = U^2 \sin \theta \cos \theta \left( m_{22} - m_{11} \right)
\]

Therefore, \( M_3 > 0 \) for \( 0 < \theta < \pi/2 \) (‘Bow up’). Therefore, a submarine under forward motion is unstable in pitch (yaw). For example, a small bow-up tends to grow with time, and control surfaces are needed as shown in the following figure.
• Restoring moment $\approx (\rho g H) \sin \theta$.

• critical speed $U_{cr}$ given by:

\[
(\rho g H) \sin \theta \geq U_{cr}^2 \sin \theta \cos \theta (m_{22} - m_{11})
\]

Usually $m_{22} \gg m_{11}, m_{22} \approx \rho g H$. For small $\theta, \cos \theta \approx 1$. So, $U_{cr}^2 \leq gH$ or $F_{cr} \equiv \frac{U_{cr}}{\sqrt{gH}} \leq 1$. Otherwise, control fins are required.