1) \( S_y^2(\omega) \)

Note: "+" was inadvertently omitted from assignment. Okay if you treated it as either \( S_y^2(\omega) \) or \( S_y(\omega) \) as long as it was treated correctly.

a) \textbf{Variance} = \( \sigma_y^2 = \sum_{i=1}^{5} S_{y_i}^2(\omega_i) \, d\omega \)

\[ \sigma_y^2 = (8)(\frac{1}{2}) + (12)(\frac{1}{2}) + (10)(\frac{1}{2}) + (4)(\frac{1}{2}) + (2)(\frac{1}{2}) = 18 \, m^2 \]

b) \textbf{Find the average upcrossings of the plane} \( z = 3 \, m \).

\[ \bar{\eta}(z) = \frac{1}{2\pi} \sqrt{\frac{M_0}{M_0}} e^{-\frac{3^2}{2M_0}} \]

where \textbf{Variance} = \( M_0 = 18 \, m^2 \)

\[ M_0 = \sum_{i=1}^{5} \omega_i^2 S_{y_i}^2(\omega_i) \, d\omega = (8)^2(\frac{1}{2}) + (12)^2(\frac{1}{2}) + (10)^2(\frac{1}{2}) + (4)^2(\frac{1}{2}) + (2)^2(\frac{1}{2}) = 37.5 \, m^2 \]

\[ \bar{\eta}(3) = \frac{1}{2\pi} \sqrt{\frac{37.5}{18 \, m^2}} e^{-\frac{9}{18}} \approx 0.1789 \text{ upcrossing/second} \]

c) \textbf{Find the minimum deck clearance to be flooded} \leq \text{once per hour.}

From part b) \( M_0 = 18 \, m^2 \), \( M_2 = 37.5 \, m^2 \)

\[ \bar{\eta}(h) \leq 1 \text{ upcrossing per hour} = \frac{1 \text{ upcrossing}}{3600 \text{ seconds}} \]

\[ \bar{\eta}(h) \leq 0.000278 \, s^{-1} \]

\[ \bar{\eta}(h) = \frac{1}{2\pi} \sqrt{\frac{37.5}{18 \, m^2}} e^{-\frac{h^2}{2 \cdot 18}} \leq 0.000278 \, s^{-1} \]

\[ e^{-\frac{h^2}{36}} \leq 0.001209 \]

\( h \geq 15.55 \) m
1) Given: $A^2 = 18 \text{ m}^2$, $\varepsilon = 0.6$

a) With a sea spectrum bandwidth of 0.6, use the approximation:

$$P(\eta \geq \eta_0) \approx \frac{2 \sqrt{1 - \varepsilon^2}}{1 + \sqrt{1 - \varepsilon^2}} e^{-\eta_0^2/2}$$

$$P(\eta \geq \eta_0) \approx \frac{2 \sqrt{1 - 0.6^2}}{1 + \sqrt{1 - 0.6^2}} e^{-\eta_0^2/2}$$

$$P(\eta \geq \eta_0) \approx 0.8888 e^{-\eta_0^2/2}$$

To find $P(\eta \geq 5 \text{ m})$, $A = 5 \text{ m}$, and $\eta_0$ is a nondimensionalized number.

$$\eta_0 = \frac{A}{\sqrt{A^2}} = \frac{5 \text{ m}}{\sqrt{18 \text{ m}^2}} = 1.17851$$

$$\therefore P(\eta \geq 1.17851) \approx 0.8888 e^{- (1.17851)^2/2}$$

$$P(\text{wave maxima exceeding 5 m}) \approx 44.49\%$$

b) 10 m?

$$\eta_0 = \frac{10 \text{ m}}{\sqrt{18 \text{ m}^2}} = 2.357$$

$$\therefore P(\eta \geq 2.357) \approx 0.8888 e^{- (2.357)^2/2}$$

$$P(\text{wave maxima exceeding 10 m}) \approx 55\%$$

c) Find the required deck height to have 1% chance of flooding:

$$0.01 \approx 0.8888 e^{-\eta^2/2}$$

$$0.0125 \approx e^{-\eta^2/2}$$

$$2 \times \ln(0.0125) \approx -\eta^2$$

$$\eta^2 \approx 2.99579 = \frac{A}{\sqrt{A^2}}$$

$$\therefore A \approx (2.99579) \sqrt{18 \text{ m}^2}$$

$$A \approx 12.71 \text{ m}$$
3) \( P(t) = \sum_{i=1}^{N} \frac{1}{2} F_i \cos (\omega t + \phi_i) \) 

**GAUSSIAN WITH ZERO MEAN**

a) \( (m + a_{33}) \ddot{x}(t) + (b_{33}) \dot{x}(t) + (c_{33}) x(t) = P(t) \)

where \( m = \) ship's mass 
\( a_{33} = \) added mass coefficient 
\( b_{33} = \) damping coefficient 
\( c_{33} = \) restoring coefficient

\( x(t) = \) heave motion

b) \[ H(\omega) = \frac{1}{-\omega^2 (m + a_{33}) + i \omega b_{33} + c_{33}} e^{i \phi_i} \]

c) **INPUT TO LTI SYSTEM IS GAUSSIAN WITH ZERO MEAN, THEREFORE OUTPUT IS GAUSSIAN WITH ZERO MEAN**

\[ f(t) \rightarrow \boxed{\text{LTI}} \rightarrow x(t) \]

d) **VARIANCE OF THE HEAVE IS THE SAME AS THE 0TH MOMENT OF THE SPECTRUM OF THE HEAVE** 
\( \sigma_x^2 = \int S_x(\omega) \, d\omega \)

AND FROM WIENER-KINNMEYER, 
\( S_x(\omega) = S_f(\omega) |H(\omega)|^2 \)

so \( \sigma_x^2 = \int S_f(\omega) |H(\omega)|^2 \, d\omega \) where \( H(\omega) \) is in part b).

e) **THINK OF ACCELERATION OF HEAVE AS AN LTI SYSTEM WITH HEAVE.**

\[ x(t) \rightarrow \boxed{\text{LTI}} \rightarrow \ddot{x}(t) \]

From part c), HEAVE IS GAUSSIAN WITH ZERO MEAN,

\( \therefore \) HEAVE ACCELERATION, \( \dot{x}(t) \), IS ALSO GAUSSIAN WITH ZERO MEAN.
3) \( f(t) \xrightarrow{\text{LTI}} H(\omega) \xrightarrow{} x(t) \xrightarrow{\text{LTI}} \tilde{x}(t) \)

\[ S_x(\omega) = S_p(\omega) |H(\omega)|^2 \quad S_{\tilde{x}}(\omega) = S_x(\omega) |H_1(\omega)|^2 \]

\[ S_{\tilde{x}}(\omega) = S_p(\omega) |H(\omega)|^2 |H_1(\omega)|^2 \]

where \( H(\omega) = \frac{1}{-\omega^2 (m a g_b) + i \omega b_{g_b} + c_{g_b}} e^{i \phi_1} \)

and \( H_1(\omega) = -\omega^2 \)

9) Yes, the output from an LTI system with a Gaussian input is also Gaussian.