Problem 1)

Abstract: The basic idea of this problem was to find the heave and pitch motions experienced by a ship transiting in head seas. Given the geometry and speed of the ship and the frequency of the waves, the first step was to calculate the added mass, damping and restoring coefficients of the system. Next, the magnitudes of the Froude-Krylov (F-K) excitation forces were determined. The following steps were based on the idea that the overall system is comprised of two linear systems in series. The first linear system acts upon the wave elevation input, and yields an output of the F-K forces. These forces are the input of the second linear system, which yields the ship's response motions as the output. The transfer functions for both of these systems were determined for both heave and pitch, uncoupled. The entire process, up to this point, was then repeated for a second input wave frequency. The transfer functions and encounter frequency were used to transform a given ambient wave spectrum into the spectra of heave and pitch response. The follow-on problem in HW9 will use these spectra to determine whether the ship met certain design criteria. Since two of the design criteria involved vertical velocity and acceleration, the heave and pitch spectra were transformed again, through the appropriate transfer functions, into velocity and acceleration spectra.

Given: The following parameters were given in the problem statement or are known physical constants.

\[ U := 10 \quad \text{Units are m/s} \]

\[ \rho := 1000 \quad \text{Units are kg/m}^3 \]

\[ g_W := 9.81 \quad \text{Units are m/s}^2 \]

\[ \omega_o := 0.5 \quad \text{Units are rad/s} \]

Encounter frequency and wave number were both calculated based on the wave frequency. The encounter frequency formula is in the form applying to head seas in particular.

\[ \omega_e := \omega_o + U \cdot \frac{\omega_o}{g} \quad \text{Units are rad/s} \]

\[ \omega_e = 0.755 \quad \text{Units are rad/s} \]

\[ k := \frac{\omega_o^2}{g} \qquad k = 0.025 \quad \text{Units are m}^{-1} \]
Question 1, Part a) I calculated a33 and b33 in an Excel spreadsheet and imported the numbers.

\[
\begin{pmatrix}
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95819 \\
95819 \\
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\begin{pmatrix}
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11561 \\
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\]

\[a33 := 95819 \quad b33 := 82999\]

I also calculated the values of the Added Mass, Damping and Restoring coefficients in Excel; they are imported here to continue the project in Mathcad.

\[\begin{align*}
A33 &= 3422372 \\
B33 &= 3133220 \\
A35 &= 54889379 \\
B35 &= -34223725 \\
A53 &= 54889379 \\
B53 &= 34223725 \\
A55 &= 1052948521 \\
B55 &= 1000238628
\end{align*}\]

Part b)

The restoring coefficient, C55, formula requires the vertical centers of buoyancy and gravity. The vertical center of gravity was given, but I had to calculate the vertical center of buoyancy. I did this in my Excel file by doing a vertical integration of the ship using waterplane areas at a step size of 1 m. I drew several diagrams of the geometry and discovered that the area of the waterplane, Awp(z), was equal to the breadth at midships at that depth times what I called the effective length, 40+2z, where z is measured positive up from the waterline in meters (i.e. z will be a negative number). The half breadth at each depth was found using pythagorean's theorem, HalfB := \sqrt{\left(10^2 - z^2\right)} . So the overall formula used was Awp(z)=2*HalfB*(40+2z). These waterplane areas were numerically integrated vertically to determine the underwater volume and the underwater center of buoyancy, and, in turn, C55.

\[\begin{align*}
C33 &= 7848000 \\
C35 &= 0 \\
C53 &= 0 \\
C55 &= 1456709917
\end{align*}\]
Project scenario: A naval vessel is transiting in head seas at speed, $U = 10$ m/s

Deep water.

Note: A33 (2D) is listed as a33 in this report, likewise with B.

1a) Incident wave frequency:

$$\omega_0 = \frac{1}{2\pi} \sqrt{\frac{g}{\rho}} = 0.5 \text{ rad/s}$$

Encounter frequency:

$$\omega_e = \frac{1}{2\pi} \sqrt{\frac{g}{\rho}} = 0.754842 \text{ rad/s}$$

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| Sum:    | 80                      | 160     | 0         | 26000      | 684474  | 0     | 0    | 90468051 | 629644  | 0 | 90068967 |

Here, I find the added mass and damping coefficients (3D) for the ship, $A_{ij}$ and $B_{ij}$ for $i,j=3,5$

The equations to calculate these coefficients were taken from Faltinsen page 56.

Since I have discrete values, I will use the trapezoidal rule as a substitute for integration.

$$h = \frac{5}{2} \text{ m}$$

Trapezoidal rule: $S = \frac{h}{2} \left( a33(-30) + 2a33(-25) + 2a33(-20) + \ldots + 2a33(25) + a33(30) \right)$

Since $a33(-30)=a33(30)$, I can simplify the formula to $h \sum(a33 \text{ values})$

$$A_{33} = 3422372$$

$$B_{33} = 3133220$$

$$A_{53} = 54989379$$

$$B_{53} = -34223725$$

$$A_{35} = -54989379$$

$$B_{35} = 34223725$$

$$A_{55} = 1052891521$$

$$B_{55} = 1000238628$$

1b) Find the restoring coefficients, $C_{ij}$ for $i,j=3,5$

Area of waterplane = $A_{wp} = 800 \text{ m}^2$
Formulas for restoring coefficients were taken from Faltinsen page 58.

To find $C_{55}$, I need to find $GM_L$ or $KB$.

To find $KB$, I will integrate the underwater volume vertically using waterplanes at 1 m intervals.

Based on geometry, half breadth at each waterline, $z$, is $\sqrt{10^2 - z^2}$

The area of each waterplane is then $2 \times \frac{1}{2} \times \text{HalfBreadth} \times \text{Effective Length}$

The 'Effective Length' is the length of the waterplane if you cut off the aft triangular portion and flip it over and put it with the forward triangular portion to make an equivalent rectangular Awp.

Here, $h = 1$ m

$\text{Volume (m}^3) = h \times \text{Sum F(M)} = 4916 \text{ m}^3$

$\text{M_volume (m}^4) = h^2 \times \text{Sum F(V)} = 18375 \text{ m}^4$

$KB = M \div \text{Vol} = 3.74 \text{ m}$

$z_B = -3.74 \text{ m}$
Part c) Now I have to find the F-K heave exciting forces and pitch exciting moments. Because \( \lambda \gg B(x) \), the F-K heave exciting force per unit length, at a given \( x \), can be approximated as \( \rho g B(x) \eta(x,t) \). Therefore, I will integrate this over the length of the ship to get the F-K heave exciting force. Similarly, I will integrate \( \rho g B(x) \eta(x,t) \) over the length of the ship to get the F-K surge exciting pitch.

\[
i = \sqrt{-1} \quad A_1 := 1 \quad \text{Here } A_1 \text{ is the amplitude of the wave, I've defined it as } 1 \text{ m.}
\]

\[
\eta(x,t) := A_1 \cdot \Re[\exp(i \omega t + ikx)]
\]

\[
2 \cdot \langle \eta, x, t \rangle := A_1 \cdot \left( \frac{1}{2} \cdot \exp(i \cdot \omega \cdot t + i \cdot k \cdot x) + \frac{1}{2} \cdot \exp(i \cdot \omega \cdot t - i \cdot k \cdot x) \right) \cdot \pi \cdot A(\omega)
\]

This is the Fourier Transform of \( \eta(x,t) \). I will need it later to find the transfer function.

Based on the given geometry, I defined the ship's breadth as a piecewise function along the length of the ship. Note that this is the entire breadth, not the halfbreadth.

\[
B(x) := \begin{cases} 
(x + 30) & \text{if } -30 \leq x \leq -10 \\
20 & \text{if } -10 < x \leq 10 \\
(-x + 30) & \text{if } 10 < x \leq 30
\end{cases}
\]

\[
F_3(t) := \rho \cdot g \cdot \left( \int_{-30}^{30} B(x) \eta(x,t) \, dx \right)
\]
\[ F_5(t) := \rho \cdot g \int_{-30}^{30} x \cdot B(x) \cdot \eta(x,t) \, dx \]
F3(t) and F5(t) are both sinusoidal type waves, so I can redefine them in terms of amplitude, which I will call Fmax.

\[ F3(0) = 7.432 \times 10^6 \]

\[ F3(t) := F3(0) \]

Visually Fmax occurs at t=0, so I will define it as such.

\[ F3_{\text{max}} := F3 \cdot \text{Re}\left[ e^{(i\omega \cdot t)} \right] \]

\[ 2 \cdot (F3, t) := F3_{\text{max}} \cdot \left( \frac{1}{2} \cdot \exp(i \cdot \omega \cdot e \cdot t) + \frac{1}{2} \cdot \exp(i \cdot \omega \cdot e \cdot t) \right) \cdot \pi \cdot \Delta(\omega) \]

\[ F5(6.24) = 3.204 \times 10^7 \]

Visually Fmax occurs at t=6.24, so I will define it as such.

\[ F5_{\text{max}} := F5(6.24) \]

\[ F5_{\text{max}} := F5_{\text{max}} \cdot \text{Re}\left[ e^{-\left( i \cdot \frac{\omega \cdot t}{2} \right)} \right] \]

\[ 2 \cdot (F5, t) := F5_{\text{max}} \cdot \left[ \frac{1}{2} \cdot \exp(i \cdot \left( \omega \cdot e \cdot t - \frac{\pi}{2} \right)) + \frac{1}{2} \cdot \exp\left( i \cdot \left( \omega \cdot e \cdot t - \frac{\pi}{2} \right) \right) \right] \cdot \pi \cdot \Delta(\omega) \]
Part d) Now I needed to find the transfer function for my first linear system, between \( \eta(t) \) and the F-K forces, \( F_3(t) \) and \( F_5(t) \). I evaluated \( \eta \) at \( x=0 \) to find these transfer functions.

Copied from above, here are the Fourier Transforms of \( \eta(x,t) \), \( F_3(t) \) and \( F_5(t) \).

\[
2 \cdot (\eta, x, t) := A_1 \cdot \left( \frac{1}{2} \cdot \exp(i \cdot \omega_c \cdot t + i \cdot k \cdot x) + \frac{1}{2} \cdot \exp\left(i \cdot \omega_c \cdot t + i \cdot k \cdot x\right) \right) \cdot \pi \cdot \Delta(\omega)
\]

Evaluated at \( x=0 \), the FT of \( \eta(x,t) \) is:

\[
2 \cdot (\eta, t) := A_1 \cdot \left( \frac{1}{2} \cdot \exp(i \cdot \omega_c \cdot t) + \frac{1}{2} \cdot \exp\left(i \cdot \omega_c \cdot t\right) \right) \cdot \pi \cdot \Delta(\omega)
\]

\[
2 \cdot (F_3, t) := F_{3\text{max}} \cdot \left( \frac{1}{2} \cdot \exp\left(i \cdot \omega_c \cdot t + i \cdot \frac{\pi}{2}\right) + \frac{1}{2} \cdot \exp\left(i \cdot \omega_c \cdot t + i \cdot \frac{\pi}{2}\right) \right) \cdot \pi \cdot \Delta(\omega)
\]

\[
2 \cdot (F_5, t) := F_{5\text{max}} \cdot \left( \frac{1}{2} \cdot \exp\left(i \cdot \left( \omega_c \cdot t - \frac{\pi}{2}\right) \right) + \frac{1}{2} \cdot \exp\left(i \cdot \left( \omega_c \cdot t - \frac{\pi}{2}\right) \right) \right) \cdot \pi \cdot \Delta(\omega)
\]

The transfer function, \( H(\omega) \), is found by dividing the output function in the frequency domain, \( F_5(\omega) \), by the input function in the frequency domain, \( \eta(\omega) \). It is easily observed that the transfer function for each heave and pitch are constant values because all other terms drop out.

\[
H_3 := \frac{F_{3\text{max}}}{A_1} \quad H_5 := \frac{F_{5\text{max}}}{A_1} \cdot e^{-\frac{\pi}{2}}
\]

where \( A_1 \) is the amplitude of the input wave elevation, \( \eta \). For a force per unit length, I will let \( A_1 = 1 \) m.

\[
H_3 = 7.432 \times 10^6 \quad H_5 = -3.204i \times 10^7
\]
Part e) In this project, the overall system includes two linear systems in series. The first system has wave elevation as input and incident force as output. The transfer function for this system was determined in part d. The second linear system has the force as input and the response motion of the ship is the output. The transfer function for this system will be found in this part for the uncoupled heave and pitch equations of motion.

The transfer function between the external force and the ship's response motion is of the now familiar form:

\[
\frac{1}{-(m + A) \cdot \omega^2 + i \cdot \omega \cdot B + C}
\]

I need to define the mass values, M33 and M55. M33 is the ship's mass, which is equal to that of the displaced water, so it is displaced volume times density. I calculated the displaced volume in my Excel file as 4916 m^3. M55 is given in the problem statement.

\[
\frac{1}{M33} = 4916\quad \text{The units are m}^3.
\]

\[
M33 := \rho \cdot V = 4.916 \times 10^6
\]

\[
M55 := 1.5 \times 10^9\quad \text{The units are kg} \cdot \text{m}^2.
\]

\[
H32(\omega) := \frac{1}{-(M33 + A33) \cdot \omega^2 + i \cdot \omega \cdot B33 + C33}\quad H32(0.755) = 2.04 \times 10^{-7} - 1.559i \times 10^{-7}
\]

\[
H32(\omega) \rightarrow \frac{1}{-8338372 \cdot \omega^2 + 3133220 \cdot i \cdot \omega + 7848000}
\]

\[
|H32(0.755)| = 2.567 \times 10^{-7}
\]

\[
H52(\omega) := \frac{1}{-(M55 + A55) \cdot \omega^2 + i \cdot \omega \cdot B55 + C55}\quad H52(0.755) = 2.536 \times 10^{-12} - 1.324i \times 10^{-9}
\]

\[
H52(\omega) \rightarrow \frac{1}{-2552981521.0 \cdot \omega^2 + 1000238628 \cdot i \cdot \omega + 1456709517}
\]

\[
|H52(0.755)| = 1.324 \times 10^{-9}
\]
Part f) I redid parts a through e in another Mathcad file and imported the relevant transfer functions under the names $H_{31b}$, $H_{51b}$, $H_{32b}$ and $H_{52b}$.

\[ H_{31b} := 5.901 \times 10^6 \quad H_{51b} := -1 \cdot 6.118 \times 10^7 \]

\[ H_{32b}(\omega) := \frac{1}{-8492507 \cdot \omega^2 + 1353808 \cdot i \cdot \omega + 7848000} \]

\[ H_{32b}(1.323) = -1.338 \times 10^{-7} - 3.415i \times 10^{-8} \quad |H_{32b}(1.323)| = 1.381 \times 10^{-7} \]

\[ H_{52b}(\omega) := \frac{1}{-2102285755.9 \cdot \omega^2 + 337862269 \cdot i \cdot \omega + 1456709517} \]

\[ H_{52b}(1.323) = -4.324 \times 10^{-10} - 8.694i \times 10^{-11} \quad |H_{52b}(1.323)| = 4.41 \times 10^{-10} \]
Part f) Here I am repeating parts a through e using \( \omega_o = 0.75 \text{ rad/s} \).

\[
\begin{align*}
U & := 10 \quad \text{Units are m/s} \\
\phi & := 1000 \quad \text{Units are kbm}^3 \\
\alpha & := 9.81 \quad \text{Units are m/s}^2 \\
\omega_o & := 0.75 \quad \text{Units are rad/s} \\
\omega_c & := \omega_o + U \frac{\omega_o^2}{g} \quad \text{Units are rad/s} \\
\omega_c & = 1.323 \quad \text{Units are rad/s} \\
\end{align*}
\]

\[
\begin{align*}
k & := \frac{\omega_o^2}{g} \quad k = 0.057 \quad \text{Units are m}^3 \text{-1} \\
\end{align*}
\]

Note, I calculated \( a_{33} \) and \( b_{33} \) in an Excel spreadsheet and imported the numbers.

\[
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108385
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\begin{pmatrix}
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29103
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\[
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b_{33} := 29103 \\
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\end{pmatrix}
\]
A naval vessel is transiting in head seas at speed, $U = 10$ m/s
Deep water.

Note: A33 (2D) is listed as a33 in this report, likewise with B.

### 1a) Incident wave frequency:

$$\omega_o = 0.75 \text{ rad/s}$$

### Encounter frequency:

$$\omega_e = 1.3233945 \text{ rad/s}$$

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**Sums:**

|          | 80 | 140 | 0 | 26000 | 715304 | 0 | 79614830 | 270762 | 0 | 52112497 |

Here, I find the added mass and damping coefficients (3D) for the ship, Aij and Bij for i=j=3,5

The equations to calculate these coefficients were taken from Faltinsen page 56.

Since I have discrete values, I will use the trapezoidal rule as a substitute for integration.

$$h = 5 \text{ m}$$

Trapezoidal rule: $0.5 \times \sum_{i=1}^{n} (a_{33}(i-30) + 2a_{33}(i-25) + 2a_{33}(i-20) + \ldots + 2a_{33}(25) + a_{33}(30))$

Since $a_{33}(30) = a_{33}(30)$, I can simplify the formula to $h \times \sum(a_{33} \text{ values})$

$$A_{33} = 3576507$$
$$B_{33} = 1355808$$
$$A_{35} = 7723979$$
$$B_{35} = -35765069$$
$$A_{53} = -7723979$$
$$B_{53} = 35765069$$
$$A_{55} = 602285755$$
$$B_{55} = 337862269$$
Note: The restoring coefficients are not frequency dependent, so these are the same as for $\omega_o=0.5$ rad/s.

1b) Find the restoring coefficients, $C_{ij}$ for $i,j=3,5$

Area of waterplane = $A_{wp} = 800 \text{ m}^2$

Formulas for restoring coefficients were taken from Faltinsen page 58.

To find $C_{55}$, I need to find $GM_L$ or $KB$.

To find $KB$, I will integrate the underwater volume vertically using waterplanes at 1 m intervals.

$z_g = -7.5$

The area of each waterplane is then $2\cdot \text{Half Breadth} \cdot (\text{which is a fn of z})$ "Effective Length".

$C_{33} = 7848000$

Based on geometry, half breadth at each waterline, $z$, is $\sqrt{10^2 - z^2}$.

$C_{35} = C_{53} = 0$

The area of each waterplane is then $2\cdot \text{Half Breadth} \cdot (\text{which is a fn of z})$ "Effective Length".

$C_{55} = 1456709517$

The "Effective Length" is the length of the waterplane if you cut off the aft triangular portion and flip it over and put it with the forward triangular portion to make an equivalent rectangular $A_{wp}$.

<table>
<thead>
<tr>
<th>$z$</th>
<th>Midships</th>
<th>&quot;Effective Awp&quot; (m)</th>
<th>$A_{wp}(z)$</th>
<th>Trapezoidal</th>
<th>Multiplier</th>
<th>$F(M)$</th>
<th>Lever Arm</th>
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<td></td>
<td></td>
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<td>18375.09</td>
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</table>

Here, $h = 1 \text{ m}$

Volume (m$^3$) = $h \cdot \text{Sum } F(M)$ = $4916 \text{ m}^3$

$M_{\text{volume}}$ (m$^4$) = $h^2 \cdot \text{Sum } F(V)$ = $18375 \text{ m}^4$

$KB = \frac{M_{\text{vol}}}{Vol} = 3.74 \text{ m}$

$z_B = -3.74 \text{ m}$
I also calculated the values of the Added Mass, Damping and Restoring coefficients in Excel; they are imported here to continue the project in Mathcad.

\[
\begin{align*}
A33 &= 3576507 \\
B33 &= 1353808 \\
A35 &= 7729979 \\
B35 &= -35765069 \\
A53 &= -7729979 \\
B53 &= 35765069 \\
A55 &= 602285755 \\
B55 &= 337862269 \\
\end{align*}
\]

\[
\begin{align*}
C33 &= 7848000 \\
C35 &= C53 = 0 \\
C55 &= 1456709517 \\
\end{align*}
\]

Part c) Now I have to find the F-K heave exciting forces and pitch exciting moments. Because \(\lambda > B(x)\), the F-K heave exciting force per unit length, at a given \(x\), can be approximated as \(pgB(x)\eta(x,t)\). Therefore, I will integrate this over the length of the ship to get the F-K heave exciting force. Similarly, I will integrate \(xpB(x)\eta(x,t)\) over the length of the ship to get the F-K surge exciting pitch.

\[i := \sqrt{-1} \quad A1 := 1 \quad \text{Here } A1 \text{ is the amplitude of the wave, I've defined it as 1 m.} \]

\[
\eta(x,t) := A1 \cdot R_d e^{(i \omega t - 0.1 + ikx)}
\]

\[\eta(0,t) = \frac{1}{2} \cdot \exp(75 \cdot i \cdot t) + \frac{1}{2} \cdot \exp(0.75 \cdot i \cdot t)\]

\[2 \cdot (\eta, x, t) := A1 \cdot \left( \frac{1}{2} \cdot \exp(\omega \cdot t - 0.1 + i \cdot k \cdot x) + \frac{1}{2} \cdot \exp(\omega \cdot t + 0.4 + i \cdot k \cdot x) \right) \cdot \pi \cdot \Delta(t)
\]

This is the Fourier Transform of \(\eta(x,t)\).

\[B(x) := \begin{cases} 
(x + 30) & \text{if } -30 \leq x \leq -10 \\
20 & \text{if } -10 < x \leq 10 \\
(-x + 30) & \text{if } 10 < x \leq 30
\end{cases}
\]
\[ F_3(t) := \rho \cdot g \cdot \left( \int_{-\frac{30}{3}}^{\frac{30}{3}} B(x) \eta(x, t) \, dx \right) \]
\[ F_5(t) := \rho \cdot g \int_{-30}^{30} x \cdot B(x) \cdot \eta(x,t) \, dx \]

Based on visual observation, I will redefine \( F_3(t) \) and \( F_5(t) \) in terms of \( F_{max} \).
\( F_3(0) = 5.901 \times 10^6 \)

\( F_{3\text{max}} := F_3(0) \)

\[ F_{3\text{max}} := F_3(0) \cdot R \left[ e^{i \omega_0 t} \right] \]

2. \( (F_3, t) := F_{3\text{max}} \cdot \left( \frac{1}{2} \cdot \exp(i \cdot \omega_0 \cdot t) + \frac{1}{2} \cdot \exp(i \cdot \omega_0 \cdot t - \pi) \right) \cdot \pi \cdot \Delta(\omega) \)

\( F_5(-3.14) = 4.331 \times 10^7 \)

\( F_{5\text{max}} := F_5(-2.1) \)

\[ F_{5\text{max}} := F_5(-2.1) \cdot R \left[ e^{i \omega_0 t} \right] \]

2. \( (F_5, t) := F_{5\text{max}} \cdot \left( \frac{1}{2} \cdot \exp(i \cdot \omega_0 \cdot t - \frac{\pi}{2}) + \frac{1}{2} \cdot \exp(i \cdot \omega_0 \cdot t - \frac{\pi}{2} + \pi) \right) \cdot \pi \cdot \Delta(\omega) \)

Part d) Now I need to find the transfer function between \( \eta(t) \) (I'm letting \( x=0 \)) and \( F_3(t) \) and \( F_5(t) \) respectively.

Copied from above, here are the Fourier Transforms of \( \eta(x,t) \), \( F_3(t) \) and \( F_5(t) \).

2. \( \eta(x,t) := A_1 \cdot \left( \frac{1}{2} \cdot \exp(i \cdot \omega_0 \cdot t + i \cdot k \cdot x) + \frac{1}{2} \cdot \exp((i \cdot \omega_0 \cdot t + i \cdot k \cdot x)) \right) \cdot \pi \cdot \Delta(\omega) \)

Evaluated at \( x=0 \), the FT of \( \eta(x,t) \) is:

2. \( \eta(t) := A_1 \cdot \left( \frac{1}{2} \cdot \exp(i \cdot \omega_0 \cdot t) + \frac{1}{2} \cdot \exp((i \cdot \omega_0 \cdot t)) \right) \cdot \pi \cdot \Delta(\omega) \)

2. \( (F_3, t) := F_{3\text{max}} \cdot \left( \frac{1}{2} \cdot \exp(i \cdot \omega_0 \cdot t) - \frac{1}{2} \cdot \exp((i \cdot \omega_0 \cdot t)) \right) \cdot \pi \cdot \Delta(\omega) \)

2. \( (F_5, t) := F_{5\text{max}} \cdot \left( \frac{1}{2} \cdot \exp(i \cdot \omega_0 \cdot t - \frac{\pi}{2}) + \frac{1}{2} \cdot \exp(i \cdot \omega_0 \cdot t - \frac{\pi}{2} + \pi) \right) \cdot \pi \cdot \Delta(\omega) \)

The transfer function \( H(\omega) \), is found by dividing the output function in the frequency domain, \( F_5(\omega) \), by the input function in the frequency domain, \( \eta(\omega) \). It is easily observed that the transfer function for each heave and pitch are constant values.
\[ H_{31} = \frac{v_{3\text{max}}}{A_1} \quad \text{and} \quad H_{51} = -\frac{v_{5\text{max}}}{A_1} \cdot e^{-\frac{\beta}{2}} \]

where \( A_1 \) is the amplitude of the input wave elevation, \( \eta \). For a force per unit length, \( f \) will let \( A_1 = 1 \) m.

\[ H_{31} = 5.901 \times 10^6 \quad \quad H_{51} = -6.118i \times 10^7 \]

Part e) In this problem the overall system includes two linear systems in series. The first system has wave elevation as input and incident force as output. The transfer function for this system was determined in part d. The second linear system has the force as input and the response motion of the ship is the output. The transfer function for this system will be found in this part for the uncoupled heave and pitch equations of motion.

The transfer function between the external force and the ship's response motion is of the form:

\[
\frac{1}{\left[-(m + A) \cdot \omega^2 + i \cdot \omega \cdot B + C\right]}.
\]

I need to define the mass values, \( M_{33} \) and \( M_{55} \). \( M_{33} \) is the ship's mass, which is equal to that of the displaced water. I calculated the displaced volume in my Excel file as 4916 m\(^3\). \( M_{55} \) is given in the problem statement.

\[ M_{33} = 4916 \text{ m}^3 \quad \text{The units are m}^3. \]

\[ M_{33} = \rho \cdot V \quad \quad M_{55} = 1.5 \times 10^6 \text{ m}^3 \quad \text{The units are kg} \cdot \text{m}^2. \]

\[
H_{32}(\omega) = \frac{1}{\left[-(M_{33} + A33) \cdot \omega^2 + i \cdot \omega \cdot B33 + C33\right]}
\]

\[
H_{32}(\omega) \rightarrow \frac{1}{-8492507 \cdot \omega^2 + 1353808 \cdot i \cdot \omega + 7848000}
\]

\[
H_{52}(\omega) = \frac{1}{\left[-(M_{55} + A55) \cdot \omega^2 + i \cdot \omega \cdot B55 + C55\right]}
\]

\[
H_{52}(\omega) \rightarrow \frac{1}{-210225755.0 \cdot \omega^2 + 337862269 \cdot i \cdot \omega + 1456709517}
\]
Part g) Now I will transform the given wave spectrum, $S(\omega_o)$, into a wave spectrum with respect to encounter frequency, $S(\omega_e)$, using the equation given in the encounter frequency lecture. Then I will use the Weiner-Kinchine relations to determine and plot the spectra of heave and pitch responses.

Because the spectrum is comprised of unit impulses, I defined $\omega_o$ in very small steps. Otherwise it distorts the shape of the spectrum.

$S(\omega_e) := \begin{cases} 2 & \text{if } \omega_o = 0.5 \\ 1.6 & \text{if } \omega_o = 0.75 \\ 0 & \text{otherwise} \end{cases}$

$S(\omega_o)$ is the ambient wave spectrum given in the problem.

$S(0.5) = 2, \quad S(0.75) = 1.6$

Once again I defined $\omega_e$ in small intervals so as not to distort the shape of the spectrum.

$S(\omega_e) := \begin{cases} 2 & \text{if } \omega_e = 0.755 \\ 1.6 & \text{if } \omega_e = 1.323 \\ 0 & \text{otherwise} \end{cases}$
Apply the Wiener-Kinchine relations for heave:

\[ H_3(\omega_c) := \begin{cases} \left( |H_3| \right)^2 \cdot \left( |H_3(0.755)| \right)^2 & \text{if } \omega_c = 0.755 \\ \left( |H_3(1)| \right)^2 \cdot \left( |H_3(1.323)| \right)^2 & \text{if } \omega_c = 1.323 \\ 0 & \text{otherwise} \end{cases} \]

\[ S_3(\omega_c) := S(\omega_c) \cdot H_3(\omega_c) \]
Now apply the Weiner-Kinchine relations for pitch:

\[
\mathcal{H}\mathcal{S}(\omega_c) := \begin{cases} 
( |H51| )^2 \cdot ( |H52(0.755)| )^2 & \text{if } \omega_c = 0.755 \\
( |H51b| )^2 \cdot ( |H52b(1.323)| )^2 & \text{if } \omega_c = 1.323 \\
0 & \text{otherwise}
\end{cases}
\]

\[
SS(\omega_c) := S(\omega_c) \cdot H\mathcal{S}(\omega_c)
\]

---

Spectrum of Pitch Motion

![Spectrum of Pitch Motion](image)

---

\[
0 \quad 0.5 \quad 1 \quad 1.5 \quad 2
\]

\[
0 \quad 0.001 \quad 0.002 \quad 0.003 \quad 0.004
\]
Problem 2)

a) SURGE EXCITATION FORCE:
\[ dF_i = (\rho \frac{\pi d^2}{4} + A_{ii}) \frac{\partial u}{\partial t} \bigg|_{x=0} dz \]
\[ \frac{\partial u}{\partial t} (x=0, z, t) = \omega^2 A e^{kz} \sin \omega t \]
\[ F_i(t) = \int_{-T}^{0} (\rho \frac{\pi d^2}{4} + A_{ii}) \omega^2 A e^{kz} \sin \omega t \, dz \]
\[ = (\rho \frac{\pi d^2}{4} + A_{ii}) \omega^2 A \sin \omega t \left[ \frac{1}{k} (1 - e^{-kt}) \right] \]

PITCH EXCITATION MOMENT:
\[ F_b(t) = \int_{-T}^{0} (-z) dF_i = (\rho \frac{\pi d^2}{4} + A_{ii}) \omega^2 A \sin \omega t \int_{-T}^{0} (-z) e^{kz} \, dz \]
\[ = (\rho \frac{\pi d^2}{4} + A_{ii}) \omega^2 A \sin \omega t \left[ \frac{1}{k} z e^{kz} - \frac{1}{k^2} e^{kz} \right]_{-T}^{0} \]
\[ = -(\rho \frac{\pi d^2}{4} + A_{ii}) \omega^2 A \sin \omega t \left[ -\frac{1}{k^2} + \frac{T}{k} e^{-kT} + \frac{1}{k^2} e^{-kT} \right] \]

b) \[ A_{ii} = \int_{-T}^{0} a_{ii} \, dz = \rho \frac{\pi d^2}{4} T \]
\[ A_{33} = \frac{1}{3} \rho \frac{\pi d^3}{8} \quad A_{55} = \int_{-T}^{0} a_{ii} z^2 \, dz = \rho \frac{\pi d^2}{4} \frac{T^3}{3} \]
\[ A_{19} = A_{51} = \int_{-T}^{0} a_{ii} z \, dz = \rho \frac{\pi d^2}{4} \frac{T^2}{2} \]
\[ A_{31} = A_{35} = A_{53} = C_{ii} = C_{13} = C_{31} = C_{15} = C_{51} = C_{35} = C_{53} = 0 \]
\[ C_{35} \rho g A_{np} = \rho g \frac{\pi d^2}{4} \]
\[ C_{95} = \rho g \int (z e^{-z^2}) \, dz + \rho g \sum_{A_{np}} x^2 \, ds = \rho g \frac{1}{2} \left[ -\frac{T}{2} - (\frac{3T}{4}) + \rho g \frac{\pi d^2}{64} \right] \]
Problem 2 cont.

c) \[ \omega_n^{(3)} = \frac{C_{33}}{m + A_{33}} = \frac{\rho g \pi d^4}{m + \frac{2}{3} \rho \pi d^3} \]

d) \[ \sum_{j=1,3,5} \left[ (M_{ij} + A_{ij}) \ddot{x}_j + B_{ij} \dot{x}_j + C_{ij} x_i \right] = F_\ell (L) \] for \( i = 1, 3, 5 \)

e) \( A_{ij} \) are the same as calculated in part b.

\[ C_{13} = C_{31} = C_{35} = C_{53} = 0 \]

\[ C_{11} = k_{11}, \quad C_{15} = k_{15}, \quad C_{51} = k_{51} \]

\[ C_{33} = k_{33} + \rho g \frac{\pi d^2}{4} \]

\[ C_{55} = k_{55} + \rho g \frac{\pi d^2}{4} \]

f) \[ \frac{F}{L} \]

The restoring force due to the cable is \( F_i = k_{ii} L \Theta = P \sin \Theta \)

For small \( \Theta \), \( \sin \Theta \approx \Theta \)

so \[ k_{ii} L \Theta \approx \Theta \]

\[ k_{ii} \approx \frac{P}{L} \]

\[ \omega_n^2 = \frac{k_{ii}}{m + \lambda_{ii}} = -\frac{P}{L (m + \rho \pi d^3)} \]