13.022 - FAll 99
QUIZ #1 - SOLUTIONS

1.

\[ S(\omega) \text{ (m}^2\text{s}) \]

\[ S = 20 \]

\[ \int_{0}^{\infty} d\omega \ S(\omega) = 2 \Omega \ S = 2 \times 0.2 \times 20 \ \text{m}^2 = 8 \ \text{m}^2 \]

\[ \sigma = 2.82 \ \text{m} \]

b)

\[ s(\omega) \]

\[ s = \frac{s}{2} \]

\[ \Delta \Omega = \Omega \]

\[ \omega_1 = \frac{3 \Omega}{2} , \ \omega_2 = \frac{5 \Omega}{2} , \ \omega_3 = \frac{7 \Omega}{2} \]

\[ S(\omega_1) = \frac{s}{2} , \ S(\omega_2) = s , \ S(\omega_3) = \frac{s}{2} \]
\[ \sum(x,t) = \sum_{i=1}^{3} A_i \cos(k_i x - \omega_i t + \phi_i) \]

- \[ \frac{1}{2} A_i^2 = S(\omega_i) \Delta \Omega \Rightarrow A_i = \left\{ \frac{2 \Delta \Omega S(\omega_i)}{2} \right\}^{1/2} \]
- \[ k_i = \frac{\omega_i^2}{g} \]
- \[ \phi_i : \text{RANDOM PHASE ANGLE \text{ UNIFORMLY DISTRIBUTED IN } (0, 2\pi)} \]

\[ f \rightarrow \phi \]
\[ 2\pi \rightarrow \varphi \]

HENCE:

\[ \sum(x,t) = \sum_{i=1}^{3} \left\{ \frac{2 \Delta \Omega S(\omega_i)}{A_i} \right\}^{1/2} \cos(k_i x - \omega_i t + \phi_i) \]

\[ (D) = -\rho \frac{\partial \phi}{\partial t} \]

\[ \text{LINEAR TERM ONLY - DROP QUADRATIC + HYDROSTATIC) } \]

\[ \phi = \sum_{i=1}^{3} \text{Re} \left\{ \frac{i g A_i}{\omega_j} e^{k_j z - i k_j x + i \omega_j t - \phi_j} \right\} \]

WAVE ELEVATION DEFINED ABOVE. CHECK USING:

\[ \sum = -\frac{1}{g} \frac{\partial \phi}{\partial t} \bigg|_{z=0} \]
d) \[ \phi = \sum_j \text{Re} \left\{ \frac{igA_j}{w_j} e^{k_j z - i k_j x + i w_j t - i \phi_j} \right\} \]

**Energy Flux Across Plane at x=0**

\[
\frac{dE}{dt} = \rho \int_{-\infty}^{0} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} \, dx , \quad \text{H.H., Eq. (175)}
\]

\[
\frac{\partial \phi}{\partial t} = \sum_j \text{Re} \left\{ -gA_j e^{k_j z + i w_j t - i \phi_j} \right\} \bigg|_{x=0}
\]

\[
\frac{\partial \phi}{\partial x} = \sum_j \text{Re} \left\{ \omega_j A_j e^{k_j z + i w_j t - i \phi_j} \right\} \bigg|_{x=0}
\]

\[
\int_{-\infty}^{0} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} \, dx = \sum_j \text{Re} \left\{ -gA_j e^{k_j z + i w_j t - i \phi_j} \right\} \bigg|_{x=0} \sum_k \text{Re} \left\{ \omega_k A_k e^{k_k z + i w_k t - i \phi_k} \right\}
\]

When mean time value is taken cross products with frequencies \( \omega_j \neq \omega_k \) average out since \( \frac{\partial}{\partial t} (\omega_j + \omega_k) = 0 \), \( \omega_j \neq \omega_k \). Hence only the diagonal terms survive leading to an energy flux

\[
\frac{dE}{dt} = \frac{1}{2} \rho g |A_1|^2 V_{g1} + \frac{1}{2} \rho s |A_2|^2 V_{g2} + \frac{1}{2} \rho s |A_3|^2 V_{g3}
\]

\[ V_{g1} = \frac{g}{2w_1}, \quad V_{g2} = \frac{g}{2w_2}, \quad V_{g3} = \frac{g}{2w_3} \]
e) When propagating into water of finite depth the amplitudes of the wave components \( (A_i) \) obey the energy conservation relation:

\[
\frac{1}{2} \rho g A_i^2 \frac{g}{2 \omega_i} = \frac{1}{2} \rho g A_{iH}^2 \nu_{gH}
\]

\[
\nu_{gH} = \left( \frac{1}{2} + \frac{K_{iH} H}{\sinh 2 K_{iH} H} \right) \frac{\omega_i}{K_{iH}}
\]

where: \( \omega_i^2 = g K_i + \tan h K_i H \rightarrow K_i \)

It follows from above identity and homework/recitation that \( A_{iH} > A_{i\infty} \) moreover this increase in amplitude is proportionally greater the lower the frequency \( \omega_i \) or the longer the waves. So qualitatively the finite-water-depth spectrum is of the form, keeping \( S \) invariant.
a) Frequencies are constant by virtue of
b) Linearity

\[
P_i = \frac{1}{2} \rho g A^2 \frac{g}{2w},
\]

\[
PT = \frac{1}{2} \rho g |A_T|^2 \frac{g}{2w},
\]

\[
PR = \frac{1}{2} \rho g |AR|^2 \frac{g}{2w},
\]

From energy balance of waves propagating
in opposite directions

\[
\sum_{A_i} \sum_{AR} ; \text{Net energy flux towards}
\]

\[
\text{The screen}
\]

\[
\frac{1}{2} \rho g \left( A^2 - |AR|^2 \right)
\]

Net energy flux past the screen

\[
\frac{1}{2} \rho g |AT|^2
\]
IN THE ABSENCE OF ENERGY DISSIPATION:

\[ |A|^2 = |A_R|^2 + |A_T|^2 \]

WITH 10% ENERGY LOSS THE ENERGY BALANCE EQUATION BECOMES

\[ E_{\text{IN-\text{NET}}} = \frac{1}{2} \rho g (|A|^2 - |A_R|^2) \]

\[ E_{\text{TRANSMITTER}} = \frac{1}{2} \rho g |A_T|^2 \]

\[ E_{\text{LOSS}} = 0.1 \frac{1}{2} \rho g |A|^2 \]

\[ E_{\text{IN-\text{NET}}} = E_{\text{TRANSMITTER}} + E_{\text{LOSS}} \]

\[ \frac{1}{2} \rho g (|A|^2 - |A_R|^2) = \frac{1}{2} \rho g (|A_T|^2 + 0.1 A^2) \]

OR \[ 0.9 |A|^2 = |A_R|^2 + |A_T|^2 \]

NO FURTHER DETERMINATION OF \( A_R \) AND \( A_T \)

IS POSSIBLE IF ADDITIONAL INFORMATION IS NOT AVAILABLE. FOR QUESTIONS c-e)

ASSUME THAT MODULUS & PHASE OF \( A_T \) AND \( A_R \) ARE KNOWN.
C) VELOCITY POTENTIAL $x < 0$:

$$
\phi^- = \text{Re} \left\{ \frac{iS_A}{\omega} e^{k_2-ikx+i\omega t} \right\} \\
+ \text{Re} \left\{ \frac{iS_A R}{\omega} e^{k_2+ikx+i\omega t} \right\}
$$

$X > 0$: $\phi^+ = \text{Re} \left\{ \frac{iS_A R}{\omega} e^{k_2-ikx+i\omega t} \right\}$

HORIZONTAL LINEAR FORCE ON SCREEN

$$
F_x = \int_{-\infty}^{0} (p^- - p^+) \, dz \times 0.9
$$

$X = 0$ \( \overline{\phi_{owt}} \).

$$
p^- = -p \frac{\partial \phi^-}{\partial t} = \rho g \text{Re} \left\{ (A + A_R) e^{k_2+i\omega t} \right\}
$$

$$
p^+ = -p \frac{\partial \phi^+}{\partial t} = \rho g \text{Re} \left\{ A_T e^{k_2+i\omega t} \right\}
$$

$$
F_x = 0.9 \rho g \frac{1}{K} \text{Re} \left\{ (A + A_R - A_T) e^{i\omega t} \right\}
$$

$$
\int_{-\infty}^{0} e^{k_2} dz \left| A + A_R - A_T \right| e^{i\psi}
$$

$$
F_x = \frac{0.9 \rho g}{K} \left| A + A_R - A_T \right| \cos (\omega t + \psi) \cdot \frac{1}{K = \omega^2/g},
$$
\[ \phi^- = \Re \left\{ \frac{\text{i} A}{\omega} e^{kz - i k x + i \omega t} \right\} + \Re \left\{ \frac{\text{i} A \Re}{\omega} e^{kz + i k x + i \omega t} \right\} \]

\[ \phi^-_x = \Re \left\{ \omega A e^{kz - i k x + i \omega t} \right\} + \Re \left\{ -\omega A \Re e^{kz + i k x + i \omega t} \right\} \]

For simplicity, set \( x = 0 \) since phase differences are accounted for uniformly in \( x \). Repeat for \( z \)-derivative.

\[ \phi^-_z = \Re \left\{ \text{i} \omega A e^{kz - i k x + i \omega t} \right\} + \Re \left\{ \text{i} \omega A \Re e^{kz - i k x + i \omega t} \right\} \]

At \( x = 0 \):
\[ \phi^-_x = e^{k^2 \omega} \Re \left\{ (A - A \Re) e^{i \omega t} \right\} \]
\[ \phi^-_z = e^{k^2 \omega} \Re \left\{ i (A + A \Re) e^{i \omega t} \right\} \]

Note that \( A = \text{real} \), \( A \Re = |A| e^{i \phi} \)

If \( A \Re = 0 \), trajectories are circular since horizontal velocity is \( 90^\circ \) out of phase from vertical velocity. Note that \( i = e^{i \pi / 2} \).
\[ \text{LET: } A - Ar = |A - Ar| e^{i\phi_1} = M_1 e^{i\phi_1} \]
\[ i(A + Ar) = |A + Ar| e^{i\phi_2} = M_2 e^{i\phi_2} \]

\[ \Rightarrow u = \phi_x = e^{k^2} \omega M_1 \cos(\omega t + \phi_1) = \frac{d\xi}{dt} \]
\[ v = \phi_z = e^{k^2} \omega M_2 \cos(\omega t + \phi_2) = \frac{d\xi}{dt} \]

WHERE \( \xi(t), \zeta(t) \) ARE HORIZONTAL, VERTICAL COORDINATES OF A FLUID PARTICLE. QUALITATIVELY, THE SOLUTION OF THIS PAIR OF ORDINARY DIFFERENTIAL EQUATIONS WILL YIELD TWO SUPERIMPOSED CIRCULAR TRAJECTORIES WITH OPPOSITE DIRECTION OF ROTATION DUE TO THE TWO WAVES MOVING IN THE OPPOSITE DIRECTION WITH DIFFERENT AMPLITUDES. THUS:

\[ \begin{array}{c}
\text{A} \\
\text{\circ} \text{w} + \text{\circ} \text{w} \\
\text{\circ} \text{Ar}
\end{array} \]

e) THE ANSWERS ABOUT \( A, \Delta t \text{ AND } Ar \) ARE THE SAME SINCE THE GROUP VELOCITY IN FINITE DEPTH FACTORS OUT.
THE HORIZONTAL FORCE WILL BECOME:

\[ F_x = 0.9 \rho g \Re \left\{ (A + A_r - A_t) e^{i \omega t} \right\} \]

\[ \times \int_{-H}^{0} d_z \frac{\cosh k(z + H)}{\cosh kH} \left( 1 + \tanh kH \right) \frac{1}{k} \]

\[ \text{So } F_x = 0.9 \rho g \Re \left\{ (A + A_r - A_t) e^{i \omega t} \right\} \frac{\tanh kH}{k} \]

ASSUMING THAT THE MODULUS & PHASE OF \( A_r, A_t \) IS THE SAME (NOT NECESSARILY SO BEFORE FURTHER ANALYSIS) THE MODULUS OF THE HORIZONTAL FORCE IN FINITE DEPTH WILL BE SMALLER SINCE

\[ \frac{\tanh kH}{k} \leq \frac{1}{k} \]

NOTE THAT AS \( H \to 0 \), \( \frac{\tanh kH}{k} \to 1 \)

FROM VALUES SMALLER THAN 1