RANKINE INTEGRAL EQUATIONS FOR SHIP FLOW
PROBLEMS WITH FORWARD SPEED

- The Green integral equation extends easily to flows past ships in calm water and in waves when the free surface condition is more complex than that of the $U=0$ frequency domain problem.

- Neumann-Kelvin problem in time domain

Consider a vessel which starts from rest at $t=0$ and translates forward with constant velocity $U$ and also possibly oscillating with amplitudes $\xi_i(t)$ if ambient waves are present.
It was shown earlier that the simplest forward speed free surface condition for the forward speed problem takes the form:

\[
\begin{cases}
\left( \frac{\partial}{\partial t} - U \frac{\partial}{\partial x} \right)^2 \psi + g \frac{\partial \psi}{\partial z} = 0, \quad z = 0 \\
\frac{\partial \psi}{\partial n} = \mathbf{V}, \quad \text{on } \overline{S_B}
\end{cases}
\]

Relative to the ship frame, the normal velocity \( \mathbf{V} \) on \( \overline{S_B} \) can be of three forms:

\[
\mathbf{V}(\mathbf{x}) = \begin{cases} 
U \eta_1, \quad t > 0: \text{forward translation} \\
\eta \xi_2(l_t), \quad t > 0: \text{radiation} \\
-\frac{\partial \psi_t}{\partial n}, \quad t > 0: \text{diffraction}
\end{cases}
\]

More general free-surface conditions with space dependent coefficients arising from gradients of the double-body flow exist and are described in the literature. The steps in deriving the relevant integral equations are very similar to the ones that follow:
Wave green functions that satisfy analytically the time-domain free surface condition stated above exist and are derived in W & L. Their evaluation is however time-consuming and they apply only to the Neumann-Kelvin formulation.

Proceeding with the derivation of the green integral equation as above and using the Rankine source as the green function:

\[ q_2(x) = -\frac{1}{4\pi} \frac{1}{|x - \xi|} \]

\[ = G(x; \xi) \]

we obtain:

\[ \frac{1}{2} \psi(\xi) + \iint_{SB} \psi(x) \frac{\partial G(x; \xi)}{\partial n_x} \, ds_x \]

\[ + \int_{S_F(\tau = 0)} \left[ \psi(x) \frac{\partial G(x; \xi)}{\partial z} - G(x; \xi) \frac{\partial \psi}{\partial z} \right] \, dx \, dy \]

\[ = \iint_{SB} G(x; \xi) \nabla(x) \, ds_x \]
Note that the integral over the free surface \((z=0)\) does not vanish since we have not used the relevant wave Green function.

Otherwise the remaining integral over \(S_B\) retains its form. The integral over \(S_\infty\) can be shown to vanish. The proof is non-trivial and may be found in references.

Over \(z=0\), it follows from the free-surface condition:

\[
\frac{\partial \Psi}{\partial z} = -\frac{1}{g} \left( \frac{\partial}{\partial t} - \frac{u}{\partial x} \right)^2 \Psi, \quad z=0
\]

Upon substitution, the second integral over \(S_F\) becomes:

\[
I_F = \iint \left[ \Psi(x) \frac{\partial G(x;\tilde{x})}{\partial z} + \frac{1}{g} G(x;\tilde{x}) \left( \frac{\partial}{\partial t} - \frac{u}{\partial x} \right)^2 \Psi(x) \right] ds_{z=0}
\]

It follows that over \(z=0\), only values and tangential gradients of \(\Psi(x)\) are now present leading to an integro-differential equation.
\[ \frac{1}{2} \varphi(\tilde{x}) + \iint_{S_B} \varphi(x) \frac{\partial G(x; \tilde{x})}{\partial n_x} \, ds_x \]

\[ + \iint_{S_F(z=0)} \left[ \varphi(x) \frac{\partial G(x; \tilde{x})}{\partial z} + \frac{1}{\rho} G(x; \tilde{x}) \left( \frac{\partial}{\partial t} - \frac{1}{\rho} \frac{\partial}{\partial x} \right)^2 \varphi(x) \right] \, dx \, dy \]

\[ = \iint_{S_B} v(x) G(x; \tilde{x}) \, ds_x \]

**Unknown is \( \varphi(x) \) over \( S_B \) and \( S_F \). Its x-derivatives may be approximated by carefully selected numerical differentiation schemes forming a core part of Rankine panel methods, discussed below.**

**The integral over the infinite free surface \( S_F(z=0) \) is truncated at some finite distance from the ship as drawn below.**
A domain denoted by the shaded area is also introduced defined as the "beach". This is located as the outer boundary of $S_f$ and selected so that over its surface the following free surface condition is enforced:

\[
\left( \frac{\partial}{\partial t} - u \frac{\partial}{\partial x} \right)^2 \psi + g \frac{\partial \psi}{\partial z} + 2v \left( \frac{\partial}{\partial t} - u \frac{\partial}{\partial x} \right) \psi + \nu^2 \psi = 0, \quad z = 0
\]

This condition differs from the Neumann-Kelvin condition by the addition of the terms that are multiplied by the dissipative parameter $\nu(x)$ which varies from $\nu = 0$ at the inner boundary to a finite value at the outer boundary of the beach.

It can be shown that converting from the time to the frequency domain via $\frac{\partial}{\partial t} \to i\omega$, $\nu$ is the familiar Rayleigh viscosity that plays a key role in the enforcement of the radiation conditions. (see W & L)