Problem 7.12
This problem is from “Advanced Fluid Mechanics Problems” by A.H. Shapiro and A.A. Sonin

Consider an incompressible flow through a series of geometrically similar machines such as fans, pumps, hydraulic turbines, etc. If $Q$ denotes volume flow, $\omega$ rotational speed, $D$ impeller diameter, $\mu$ fluid viscosity, and $\rho$ fluid density,

(a) show that dynamic similarity requires that $Q/\omega D^3$ and $\rho Q/\mu D$ be fixed.

(b) Show that if $Q/\omega D^3$ and $\rho Q/\mu D$ are fixed in a series of tests, then $\Delta P/\rho \omega^2 D^2$ must remain constant, where $\Delta P$ is the change in head across the machine, expressed in pressure units.

(c) Find the form of the relation between the work output per unit mass of fluid $W$, and the the given variables, in a series of tests where $Q/\omega D^3$ and $\rho Q/\mu D$ are fixed.
Solution:
(a) The variables in the problem are related through \( f(Q, \omega, D, \mu, \rho, \Delta P) = 0 \) where

\[
Q \text{ [L}^3\text{T}^{-1}] : \text{Volume flow rate} \\
\omega \text{ [T}^{-1}] : \text{Rotational Speed} \\
D \text{ [L]} : \text{Impeller diameter} \\
\mu \text{ [ML}^{-1}\text{T}^{-1}] : \text{Fluid viscosity} \\
\rho \text{ [ML}^{-3}] : \text{Fluid density} \\
\Delta P \text{ [ML}^{-1}\text{T}^{-1}] : \text{Change in head across machine}
\]

As our primary variables, we pick \( \rho \) for the fluid, \( \omega \) for the flow and \( D \) for the geometry. We have

\[
n = 6 \text{ variables} \\
r = 3 \text{ primary dimensions} \\
\Rightarrow j = 6 - 3 = 3 \text{ dimensionless groups}
\]

Now,

\[
\Pi_1 = \frac{Q}{\rho^a \omega^b D^c}
\]

By inspection, we find \( a = 0, \ b = 1 \) and \( c = 3 \). Therefore

\[
\Pi_1 = \frac{Q}{\omega D^3}
\]

Similarly, we find

\[
\Pi_2 = \frac{\mu}{\rho \omega D^2}
\]

Let

\[
\Pi' = \frac{\Pi_1}{\Pi_2} = \frac{Q}{\omega D^3} \times \frac{\rho \omega D^2}{\mu}
\]

\[
\Rightarrow \Pi' = \frac{\rho Q}{\mu D}
\]

Therefore, dynamic similarity requires that

\[
\Pi_1 = \frac{Q}{\omega D^3} = C_1
\]

and \( \Pi' = \frac{\rho Q}{\mu D} = C_2 \)

where \( C_1 \) and \( C_2 \) are constants.

(b) The third non-dimensional group is given by

\[
\Pi_3 = \frac{\Delta P}{\rho \omega^2 D^2}
\]
Therefore, from the Buckingham Π theorem,

\[ \Pi_3 = f(\Pi_1, \Pi') \]  
\[ \Rightarrow \frac{\Delta P}{\rho \omega^2 D^2} = f\left(\frac{Q}{\omega D^3}, \frac{\rho Q}{\mu D}\right) \]  

(7.12j)

(7.12k)

Now if \( \Pi_1 = \frac{Q}{\omega D^3} \) and \( \Pi' = \frac{\rho Q}{\mu D} \) are constants, then \( f(\Pi_1, \Pi') = f(C_1, C_2) = C \) where \( C \) is a constant. Hence, equation (7.12k) implies

\[ \frac{\Delta P}{\rho \omega^2 D^2} = C \]  

(7.12l)

(c) The work output \( w \) is given by

\[ w = Q \Delta P \]  

(7.12m)

We know from equation (7.12h) that \( Q = C_1 \omega D^3 \) and from equation 7.12l that \( \Delta P = C \rho \omega^2 D^2 \). Substituting this into equation (7.12m), we have

\[ w = C_1 C \rho \omega^3 D^5 = K \rho \omega^3 D^5 \]  

(7.12n)

where \( K \) is a constant. Thus we have per unit mass that

\[ W = \frac{w}{\rho D^3} = KD^2 \omega^3 \]  

(7.12o)