**Symmetry of Stress Tensor**

Imagine an arbitrary fluid element which in 2-D is a rectangle with width $\delta x_1$ in the $x, 1$ direction and height $\delta x_2$ in the $y, 2$ direction; the element also has a width $\delta x_3$ in the $z, 3$ direction (Figure 1). Stresses acting on each face can be calculated using the values at point $O$ (center of the element) and applying a Taylor series expansion in each direction:

- **Shear stress acting on the right wall:** $\tau_{12} \big|_O + \delta \tau_a$
- **Shear stress acting on the left wall:** $\tau_{12} \big|_O - \delta \tau_a$
- **Shear stress acting on the bottom wall:** $\tau_{21} \big|_O - \delta \tau_b$
- **Shear stress acting on the top wall:** $\tau_{21} \big|_O + \delta \tau_b$

in which:

\[
\delta \tau_a = \frac{\partial \tau_{12}}{\partial x_1} (\delta x_1/2) \\
\delta \tau_b = \frac{\partial \tau_{21}}{\partial x_2} (\delta x_2/2)
\]

![Figure 1: Taylor series expansion for the shear stresses acting on a material element of size $\delta x_1, \delta x_2$.](image)

Knowing that the normal stresses acting on each plane will not lead to any net torque around axis $x_3$ passing through point $O$ one can calculate the net exerted torque on the element ($M_O$) by accounting for all the shear stresses acting on the element:

\[
\sum M_O = (\tau_{12} + \delta \tau_a)(\delta x_2 \delta x_3)(\delta x_1/2) + (\tau_{12} - \delta \tau_a)(\delta x_2 \delta x_3)(\delta x_1/2) \\
- (\tau_{21} + \delta \tau_b)(\delta x_1 \delta x_3)(\delta x_2/2) - (\tau_{21} - \delta \tau_b)(\delta x_1 \delta x_3)(\delta x_2/2)
\]
Which can be simplified to give:

\[ \sum M_O = (\tau_{12} - \tau_{21})\delta x_1\delta x_2\delta x_3 \]  

(1)

On the other hand we know that the following holds:

\[ \sum M_O = I\dot{\omega}_3 \]

in which \( I \) is the moment of inertia around \( x_3 \) axis passing through point \( O \) and for a cuboidal element it is:

\[ I = \frac{\rho}{12}\delta x_1\delta x_2\delta x_3(\delta x_1^2 + \delta x_2^2) \]  

(2)

Combining (1) and (2) will result in:

\[ \dot{\omega}_3 = \frac{12}{\rho} \frac{\tau_{12} - \tau_{21}}{\delta x_1^2 + \delta x_2^2} \]

It is easy to see that if one shrinks the element to a very small volume (i.e. \( \delta x_1 \) and \( \delta x_2 \rightarrow 0 \)) the rotational acceleration of the element (\( \dot{\omega}_3 \)) will diverge to infinity unless the shear stress difference also tends to zero at least as fast as \( \delta x_2 \rightarrow 0 \) (thus \( \tau_{12} - \tau_{21} = 0 \)). Since infinite rotational acceleration is not physically possible the stress tensor should be symmetric, \( \tau_{ij} = \tau_{ji} \).

\textsuperscript{1}The mentioned proof is true in the absence of magneto-hydrodynamic forces or other non-conservative body forces.
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