Problem 4.04
This problem is from “Advanced Fluid Mechanics Problems” by A.H. Shapiro and A.A. Sonin

A nozzle with exit area $A_2$ is mounted at the end of a pipe of area $A_1$, as shown. The nozzle converges gradually, and we assume that the flow in it is (i) approximately uniform over any particular station $x$, (ii) incompressible, and (iii) inviscid. Gravitational effects are, furthermore, taken as negligible. The volume flow rate in the nozzle is given as $Q$ and the ambient pressure is $p_a$.

(a) Derive an expression for the gage pressure at a station where the area is $A(x)$.

(b) Show, by integrating the $x$-component of the pressure force on the nozzle’s interior walls, that the net $x$-component of force on the nozzle due to the flow is independent of the specific nozzle contour and is given by

$$F = \rho Q^2 \frac{(A_1 - A_2)^2}{2A_1 A_2^2}$$

(c) The expression in (b) predicts that $F$ is in the positive $x$-direction regardless of whether the nozzle is converging ($A_2 < A_1$) or diverging ($A_2 > A_1$). Explain.
Solution:

Given: $Q$, $A_1$, $A_2$ are constants.

(a) By mass conservation,

$$(\text{mass in}) = \rho v_1 A_1 = \rho v(x) A(x) = (\text{mass out})$$

Since there is no change in the mass inside the CV:

$$v_1 A_1 = Q = v(x) A(x)$$

$$\Rightarrow v(x) = \frac{Q}{A(x)}$$

Apply Bernoulli’s equation along a stream line from station 1 to 2:

Note that all the assumption required for Bernoulli have been satisfied:

(a) inviscid
(b) along a streamline
(c) steady
(d) constant density
(e) no work/energy input or loss

$$p(x) + \frac{1}{2} \rho v(x)^2 = p_a + \frac{1}{2} \rho v_2^2$$

Therefore,

$$p_g(x) = p(x) - p_a = \frac{1}{2} \rho \left( v_2^2 - v^2 \right) \Rightarrow p_g(x) = \frac{1}{2} \rho Q^2 \left( \frac{1}{A_2^2} - \frac{1}{A(x)^2} \right)$$

(b) Integrate the pressure along the nozzle to obtain the $x$-component of pressure force

$$F_x = \int_{A_1}^{A_2} p_g dA = \int_{A_1}^{A_2} p_g d (\text{projected vertical area})$$

$$= \int_{A_1}^{A_2} p_g (A_1 - A) = - \int_{A_1}^{A_2} p_g dA$$
We can reverse the integration limits to get rid of the minus sign in front and substitute Eq. (4.04a):

\[ F_x = \int_{A_2}^{A_1} p_g dA \]

\[ = \rho Q^2 \int_{A_2}^{A_1} \left( \frac{1}{A_2'^2} - \frac{1}{A_1'^2} \right) dA \]

\[ \Rightarrow F_x = \frac{\rho Q^2 (A_1 - A_2)^2}{2 A_1 A_2} \]  

\[(4.04b)\]

(c)

For \((A_2 < A_1)\), \(p_g(x)\) is positive. Since the pressure is greater on the inside than the outside, the net pressure force acts on the inner wall.

For \((A_2 > A_1)\), \(p_g(x)\) is negative. Thus, the net pressure force acts on the outer wall, still pointing to the right.