Problem 4.11
This problem is from “Advanced Fluid Mechanics Problems” by A.H. Shapiro and A.A. Sonin

An incompressible, inviscid liquid flows with speed $V$ vertically downward from the nozzle of the radius $R$. The liquid density $\rho$ is high compared with that of the ambient air. The surface tension between the liquid and the air is $\sigma$.

(a) Obtain an expression which relates the local radius $r$ of the liquid stream to the distance $x$ from the nozzle.

(b) Show that for sufficiently large $x$,

$$r \approx R \left( \frac{V^2}{2gx} \right)^{\frac{1}{4}}$$  \tag{4.11a}

(c) Write down all the criteria which must be satisfied for this expression to be a good approximation. State each criterion as ‘$x$ must be very large compared with $y$’, where $y$ is some combination of the given quantities $V$, $R$, $g$, and $\sigma$. 

Solution:

- (a) Using Bernoulli between 1 and 2,

\[
\frac{1}{2} \rho V^2 + P_a = \frac{1}{2} \rho V_1^2 - \rho gx + P_a,
\]

simplifying,

\[
\frac{2}{\rho} \left( \frac{1}{2} \rho V^2 + \rho gx \right) = \frac{1}{2} \rho V_2^2 - \frac{2}{\rho},
\]

then,

\[
V^2 + 2gx = V_2^2, \Rightarrow V_2 = \sqrt{V^2 + 2gx}.
\]

Also, from mass conservation

\[
\pi R^2 V = \pi r^2 V_2, \Rightarrow \left( \frac{r}{R} \right)^2 = \frac{V}{V_2}.
\]

Finally adding the information from Bernoulli,

\[
\left( \frac{r}{R} \right)^2 = \frac{V}{\sqrt{V^2 + 2gx}}.
\]

- (b) For \(\frac{2gx}{V^2} \gg 1\) \(\Rightarrow \left( \frac{r}{R} \right)^2 \approx \left( \frac{V^2}{2gx} \right)^{\frac{1}{2}}\), \(\Rightarrow\)

\[
\frac{r}{R} = \left( \frac{V^2}{2gx} \right)^{\frac{1}{2}}.
\]

- (c) For the solution to apply,

\[
\frac{V^2}{2gx} \ll 1, \quad OR \quad \frac{\rho V^2}{2pgx} \ll 1,
\]

then,

\[
x \gg \frac{V^2}{2g}.
\]
Also, since we neglected surface tension,

\[ \frac{\Delta P_\sigma}{\Delta P_x} = \frac{\sigma}{\rho g x} \ll 1, \Rightarrow \frac{\sigma}{r \rho g x} \ll 1. \]  \hspace{1cm} (4.11j)

Now, let’s get an estimate of the order of magnitude of \( r(x) \) from (b),

\[ r \approx R \left( \frac{V^2}{2g x} \right)^{\frac{1}{4}}, \Rightarrow r \sim \frac{RV^{\frac{1}{2}}}{(gx)^{\frac{1}{4}}}, \]  \hspace{1cm} (4.11k)

now, substituting into the \( x \) requirement,

\[ \frac{\sigma (gx)^{\frac{1}{4}}}{RV^{\frac{1}{2}} \rho g x} \ll 1, \Rightarrow x^{\frac{3}{4}} \frac{\sigma}{RV^{\frac{1}{4}} \rho g^{\frac{3}{4}}}, \]  \hspace{1cm} (4.11l)

finally,

\[ x \gg \frac{\sigma^{\frac{3}{4}}}{R^{\frac{3}{4}} g^{\frac{3}{4}} V^{\frac{1}{4}}}, \]  \hspace{1cm} (4.11m)
2.25 Advanced Fluid Mechanics
Fall 2013

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