Problem 10.04
This problem is from “Advanced Fluid Mechanics Problems” by A.H. Shapiro and A.A. Sonin

A steady, inviscid, incompressible flow experiences a change of cross sections between stations (1) and (2), as shown. At station (1), the velocity distribution is

\[ v_x = U + ky, \quad -\frac{b_1}{2} < y < \frac{b_1}{2}, \quad (10.04a) \]

where \( U \) is the mean flow velocity. There are no body forces acting on the fluid. Considering \( U, k, \) and the system dimensions given, determine expressions for

- (a) the vorticity at station (1),
- (b) the vorticity at station (2),
- (c) the velocity distribution at station (2),
- (d) the ration \( \frac{\Delta v_x}{v_{av}} \) average of the total velocity excursion to the average velocity at (2), divided by the same quantity at (1).

**Answer**

\[ \frac{\Delta v}{v_{av}} = \left( \frac{A_1}{A_1} \right)^2, \quad (10.04b) \]

where \( A \) stands for \( a \cdot b. \)
Solution:

- (a) The vorticity vector at station (1) is
  \[ \omega = -\frac{\partial v_x}{\partial y} \hat{e}_z = -k \hat{e}_z. \]  
  \[ \text{(10.04c)} \]

- (b) The x and y components of \( \omega \) are zero initially. Let’s first look at how these evolve,
  \[ \frac{D\omega}{Dt} = (\omega \cdot \nabla)V, \]  
  \[ \text{(10.04d)} \]
in particular, in the x direction, (only direction not null due at the inlet and outlet)
  \[ \frac{D\omega_x}{Dt} = \omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z} \]  
  \[ \Rightarrow \frac{D\omega_x}{Dt} = \omega_x \frac{\partial v_x}{\partial z} \Rightarrow D\omega_x = \int_{\text{station}1}^{\text{station}2} \omega_z \frac{\partial v_x}{\partial z} \,Dt = 0. \]  
  \[ \text{(10.04e)} \]

Although \( \frac{D\omega}{Dt} \) is not zero always (it is not zero specifically in the region the wall bends), we can still argue that the above integral is zero. The streamlines bend across wall-bends causing pressure differential in cross-stream direction resulting in velocity differential \( \frac{\partial v_z}{\partial x} \). However, the wall bends are once concave and then convex - hence, effectively cancel each other once we integrate over the entire particle motion across the flow regime. This is a loose physical argument but we have to live with this - to escape from otherwise complicated mathematics!

\[ \Rightarrow \omega_x = \text{Const} = 0 \]

Similarly, we have the y-component:

\[ \omega_y = \text{Const} = 0 \]

Now, let’s look at the evolution of \( \omega_z \),

\[ \frac{D\omega_z}{Dt} = \omega_z \frac{\partial v_z}{\partial z}, \]  
  \[ \text{(10.04g)} \]

Replacing \( \frac{D}{Dt} \) by \( \left. \frac{d}{dt} \right|_m \) for derivative along a material particle,

\[ \left. \frac{d\omega_z}{dt} \right|_m = \omega_z \frac{\partial v_z}{\partial z} \Rightarrow \int \left. \frac{d\omega_z}{dt} \right|_m = \int \left. \frac{\partial v_z}{\partial z} \right|_m \,dt. \]  
  \[ \text{(10.04h)} \]

From the figure, we can see that the variation of cross section in z-direction (i.e. variation of a) happens first (when the cross section in y direction remains the same). Similarly, the variation in y direction cross section is independent of z variation in this problem. Equation (h) only needs to be applied in the region where the variation of cross section in z-direction happens (since \( \frac{\partial v_z}{\partial x} \) exists only in that region), i.e. from station 1 to say station 1’. From station 1 to station 1’ continuity gives:

\[ \frac{\partial v_z}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \Rightarrow \frac{\partial v_z}{\partial z} = -\frac{\partial v_z}{\partial x}, \]  
  \[ \text{(10.04i)} \]

We plug the above in (h) and do some rearrangement as below,

\[ \int_1^2 \left. \frac{d\omega_z}{\omega_z} \right|_m = \int_1^{1'} \left. \frac{d\omega_z}{\omega_z} \right|_m = -\int_1^{1'} \frac{\partial v_z}{\partial x} \,dt \left|_m = -\int_1^{1'} \frac{dt}{dx} \frac{dv_z}{dx} \,dt \left|_m = -\int_1^{1'} \frac{dv_z}{v_x} \,dt \left|_m. \right. \]  
  \[ \text{(10.04j)} \]

\[ \Rightarrow \frac{\omega_{z,2}}{\omega_{z,1}} = \frac{\omega_{z,1'}}{\omega_{z,2}} = \frac{v_{x,1}}{v_{x,1'}} = \frac{a_2}{a_1} \Rightarrow \omega_{z,2} = -\frac{a_2}{a_1} k \hat{e}_z. \]  
  \[ \text{(10.04k)} \]

Hence, the vorticity vector at station 2 is \( \omega_2 = -\frac{a_2}{a_1} k \hat{e}_z. \)
• (c) Now, at station 2,
\[ \omega_z = -\frac{a_2}{a_1} k = -\frac{\partial v_x}{\partial y}, \Rightarrow v_x = \frac{a_2}{a_1} ky + C, \]  
(10.04l)
where $C$ is a constant of integration. Mass conservation between station 1 and 2 gives
\[ a_2 \int_{\frac{1}{2}}^{b_2} \left( \frac{a_2}{a_1} ky + C \right) = a_1 \int_{-\frac{1}{2}}^{b_1} (ky + U) dy, \]  
(10.04m)
\[ \Rightarrow a_2 b_2 C = a_1 b_1 U, \Rightarrow C = \frac{a_3 b_1}{a_2 b_2} U = \frac{A_1}{A_2} U. \]  
(10.04n)
Hence, velocity distribution at station 2 is $v_x = \frac{a_2}{a_1} ky + \frac{A_1}{A_2} U$.

• (d) First, let’s calculate the requested values at 1 and 2 in order to get the ratio. First at 2, then
\[ \frac{\Delta v}{v_{av}}|_2 = \left( \frac{a_2}{a_1} k \frac{b_2}{2} + \frac{A_1}{A_2} U \right) - \left( \frac{a_2}{a_1} k \left( -\frac{b_2}{2} \right) + \frac{A_1}{A_2} U \right) = \frac{a_2 b_2 k A_2}{a_1 A_1 U} = \frac{k A_2^2}{a_1 A_1 U}. \]  
(10.04o)
And for station 1,
\[ \frac{\Delta v}{v_{av}}|_1 = \frac{k \frac{b_1}{2} + U}{U} - \frac{k \frac{b_1}{2} + U}{U} = \frac{kb_1}{U}, \]  
(10.04p)
then we can finally calculate the ratio,
\[ \frac{\Delta v/v_{av}|_2}{\Delta v/v_{av}|_1} = \left( \frac{A_2}{A_1} \right)^2. \]  
(10.04q)