Compressible Fluid Dynamics: Lect. 1

Admin: office hours? Reschedule 2 classes
Textbooks (note: syllabus is evolving)

Grading: 60% HW, class participation
40% term project
- written report + presentation
- will point out suggested topics
during lecture
- e.g. review "classic" paper and
expand (extra calculations, apply
to specific problem...)

Compressible Flows (a.k.a. Gas Dynamics)

A flow may be considered *incompressible if a material
element conserves volume as it is convected by the flow
(show multi-media movie)

More intuitively, a flow is *compressible if changes in
fluid density play an essential role. (show multi-media
movie, schlieren photography: see changes in index of refraction
due to changes in density, shock: discontinuity in density,
velocity, etc.)

Compressibility may be important in many circumstances,
most commonly, when typical velocities in the fluid
exceed the speed of sound, i.e.

\[ \text{Mach #} = \frac{V}{c} \leq 1 \]

\( V \) we will see that this is a state dependent
property (If anyone knows speed of sound in air, \( c \approx 350 \text{ m/s} \)
and is not constant,
or water, \( c \approx 1500 \text{ m/s} \), calc. \( V \) here.)
Let us now consider the problem of compressible flow of a perfect gas, with

\[ \frac{\gamma}{P/P_0} \]

Figure 5.9

\[ M \]

\[ \gamma = 1.40 \]

Pressure drop vs. velocity for a perfect gas with

\[ \frac{P - P_0}{P_0} \]

\[ n \]

\[ \frac{\gamma P_0}{n} \]

(5.14)

Joule equation

specific heats

fluid which is

compressible flow

incompressible flow
Review of Fluid Mech (chpt. 1 Thomason)

To find fundamental conservation laws, use Reynolds's Transport Theorem:

\[
\frac{\text{rate of change}}{\text{of "stuff" in } V} = \left( \text{rate of stuff being created in } V \right) + \left( \text{stuff flowing in} \right) - \left( \text{stuff flowing out} \right)
\]

"stuff" = mass, momentum, energy, entropy etc.

Mass Conservation

\[
\begin{align*}
\frac{\partial}{\partial t} \int_V \rho \, dV + \int_S \rho \mathbf{u} \cdot \mathbf{n} \, dS &= 0 \\
\text{suppose } V \text{ is fixed} \\
\text{divergence thm.} \\
\int_V \frac{\partial \rho}{\partial t} \, dV + \int_V \nabla \cdot (\rho \mathbf{u}) \, dV &= 0
\end{align*}
\]

Since \( V \) is arbitrary:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

Recall \( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla = \frac{D}{Dt} = \text{material derivative} \)

\[
\frac{D \rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0
\]

(Recall for incompressible flows, \( \frac{D \rho}{Dt} \equiv 0 \Rightarrow \nabla \cdot \mathbf{u} = 0 \))
Conservation of momentum (assume inviscid for now)

\[ \frac{1}{dt} \int_\Omega \rho \mathbf{u} \, dV + \int_{\partial \Omega} \rho \mathbf{u} \cdot \hat{n} \, ds = \text{momentum created} \]

(Forces!)

\[ = \int_\Omega \rho g \, dV + \int_{\partial \Omega} \mathbf{P} \cdot \hat{n} \, ds \]

\[ \int_\Omega \frac{\partial}{\partial t} (\rho \mathbf{u}) \, dV + \int_\Omega \nabla \cdot (\rho \mathbf{u} \mathbf{u}) \, dV = \int_\Omega \rho g \, dV - \int_\Omega \nabla P \, dV \]

\[ = \frac{\partial}{\partial x_i} (\rho u_i) \]

\[ \frac{\partial}{\partial t} (\rho \mathbf{u}) + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla (\rho \mathbf{u}) = \rho \mathbf{g} - \nabla P \]

\[ \frac{\partial}{\partial t} (\rho \mathbf{u}) + \rho \mathbf{u} \cdot \nabla \mathbf{u} = \rho \mathbf{g} - \nabla P \]

\[ \rho \frac{D \mathbf{u}}{Dt} + \mathbf{u} \left( \frac{D \rho}{Dt} + \rho \mathbf{u} \cdot \nabla \rho \right) = \rho \mathbf{g} - \nabla P \]

\[ \rho \frac{D \mathbf{u}}{Dt} = -\nabla P + \rho g \]

Euler's Equation

Note one important difference between compressible and incompressible flows already:

Incompressible Euler (3 eq) unknowns: \( \mathbf{u} \), \( p \) (4)

\( \nabla \cdot \mathbf{u} = 0 \) (1 eq)

Compressible Euler (3 eq) unknowns: \( \mathbf{u} \), \( p \), \( \rho !! \) (5!)

\( \frac{D \rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \) (1 eq)
Brief review of essential thermodynamics (Chapter 2)

A local thermodynamic state is fixed by any two thermodynamic variables. (e.g. \(p\) and \(s\); or \(p\) and \(T\), etc.) (Equilibrium statement! Never true for transport! But we fudge this.)

\[ \Rightarrow \text{Term Project paper:} \]


Lighthill "Viscosity effects in waves of finite amplitude" in Batchelor and Davies Surveys in Mechanics, Cambridge University Press (no slip cond.)

\[ \text{Internal energy: } e = e(u, s) \]

\[ \text{Enthalpy: } h = h(s, p) = e + pv \]

**First law:**

\[ de = dq + dw = Tds - pdV \]

\[ \text{Reversible: no entropy produced} \]

\[ dh = de + pdV + vdp = Tds + vdp \]

**Heat capacities:**

\[ C_v = \left. \frac{\partial e}{\partial T} \right|_v \]

\[ C_p = \left. \frac{\partial h}{\partial T} \right|_p \]

Maxwell's relations (pg. 61 in Thermen

**RTT for energy + entropy:**

\[ \rho \frac{D}{Dt} \left( e + \frac{u^2}{2} \right) = \text{div} \cdot \rho g u - \nabla \cdot \left( \frac{\rho u}{T} \right) \geq 0 \]

\[ \rho \frac{D}{Dt} + \text{div} \cdot \nabla \left( \frac{\rho u}{T} \right) \geq 0 \]

**Heat Flux:**
Clearly we need one more equation. Typically, this final equation is cons. of energy (or possibly entropy).

→ Thermodynamics!

\[ \text{Acoustics - fluid motions associated w. the propagation of sound.} \]

Start w. fluid medium at rest. \( \mathbf{u}_0 = 0 \), \( \rho_0 \), \( P_0 \).

Sound waves are small amplitude pressure fluctuations in the media. (how small is small? look up #13)

\[ \text{Pressure} \]

\[ \text{Neglect viscous dissipation; neglect heat transfer} \]

→ The flow is \( \approx \) isentropic \( (s = s_0) \)

Recall, for a pure substance, we only need two thermodynamic variables to fix the state of the system, e.g. \( P \) and \( s \): \( \rho(P, s) \) ①

Perturb about rest state: \[ \mathbf{u} = \mathbf{0} + \mathbf{u}' \]
\[ \rho = \rho_0 + \rho' \quad P = P_0 + P' \]

\[ \text{Mass:} \quad \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho = 0 \]
\[ \text{small} \quad \frac{\partial \rho'}{\partial t} + (\rho_0 \rho') \nabla \cdot \mathbf{u}' + \mathbf{u}' \cdot \nabla \rho' = 0 \]

② \[ \frac{\partial P}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = 0 \] (dropping primes)

\[ \text{Euler:} \quad \frac{(\rho + \rho')}{\partial t} \frac{\partial \mathbf{u}}{\partial t} = - \nabla P' + \mathbf{f} \]
\[ \text{both linear neglect} \]
\[ \rho_0 \frac{\partial \mathbf{u}}{\partial t} = - \nabla P \] (dropping primes)
\[ \rho_0 \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = - \nabla P \]
From (1) \[
\frac{\partial P}{\partial t} = \left( \frac{\partial P}{\partial s} \right)_s \frac{\partial s}{\partial t} + \left( \frac{\partial P}{\partial S} \right)_S \frac{\partial S}{\partial t}
\]
\[
\frac{1}{c^2} \equiv \left( \frac{\partial P}{\partial s} \right)_s
\]

From (2) \[
\frac{\partial}{\partial t} \left( \frac{\partial P}{\partial t} = -\rho \frac{c^2}{\partial x} \frac{\partial u}{\partial x} \right)
\]
(consider 1D)

(3) \[
\frac{\partial}{\partial x} \left( \rho \frac{\partial u}{\partial x} = -\frac{\partial P}{\partial x} \right)
\]

Classical Wave Equation
(Similarly for \( P \) and \( u \), etc)

Solutions to the wave equation:

\[
P = f(x - ct) + g(x + ct)
\]

shifts \( f \) right
shifts \( g \) left

Show that this is a solution: (let \( g = 0 \) ) let \( f = x - ct \)

\[
\begin{align*}
\frac{\partial P}{\partial t} &= \frac{\partial P}{\partial x} \frac{\partial x}{\partial t} = -c \frac{\partial P}{\partial x} = -c \cdot P' \quad \text{plug into wave eq.} \\
\frac{\partial P}{\partial x} &= P'
\end{align*}
\]

\[ c^2 P'' = c^2 P'' \quad \text{Note: we can interpret } c
\]
as the speed of the wave

H.W. show \( u \) and \( P \) are also governed by the wave eq.
Speed of sound in an ideal gas:

\[ P v = RT \Rightarrow P = \rho RT \]

\[ \text{specific gas const} \]

For an isentropic process:

\[ P v^y = \text{const} \Rightarrow P \rho^{-y} = \text{const} \]

\[ \gamma = \frac{C_P}{C_v} \text{ (ratio of specific heats)} \]

\[ c^2 = (\frac{\delta P}{\delta \rho})_s = \gamma \text{ const.} \rho^{-1} \Rightarrow \delta \rho^{-1} P \rho^{-y} = \frac{\gamma P}{\rho} \]

\[ c = \sqrt{\frac{\delta P}{\rho}} = \sqrt{\gamma RT} \]

\[ \gamma = 1.4 \quad P = 1 \text{ atm} \approx 10^5 \text{ N/m}^2 \quad \rho \approx 1.25 \text{ kg/m}^3 \]

\[ \Rightarrow \quad C_{\text{air}} \approx 335 \text{ m/sec} \]

Speed of sound in a liquid

Define in terms of \( k_s = \rho (\frac{\delta P}{\delta \rho})_s \) = isentropic bulk modulus

\[ c^2 = \frac{k_s}{\rho} \]

\[ C_{\text{water}} \approx 1500 \text{ m/s} \]

Sound waves in a moving medium:

E.g. \[ u \rightarrow \downarrow \text{wave} \uparrow \quad \nu = u + c \]
Traffic analogy

\[ p = \text{"density"} = \frac{\# \text{cars}}{\text{unit length}} \]

\[ \text{traffic speed} \]

\[ t \]

\[ \text{initial } p \]

\[ \text{high density} \]

\[ \text{low density} \]

\[ \text{expansion fan} \]

\[ t_{\text{later}} \]
$M < 1$ subsonic

$M > 1$ supersonic

$U \approx 1$ transonic

Both upstream and downstream flow.

"See" the disturbance upstream "sees" nothing!! Pulse cannot affect upstream conditions.

In 3D

Stationary fluid

$u \rightarrow c$ (object moving to left @ velocity $u$)

For a stationary observer, frequency is decreased (due to Doppler effect).

$u > c$ Mach cone (propagation of information)

$\sin u = \frac{v}{u} = \frac{1}{M}$

Zone of silence (observer cannot sense object)

Wave equations and causality

(Do this earlier w. wave eq.)
The shock is a discontinuity in physical quantities (p, p, u, etc.).

Supersonic flow past an object:

Low p, p

shock

High p, p

\( \theta \to 0, \rho \to \infty \) (but shock Mach wave becomes weaker)

Shock

expansion fan

envelope of Mach lines

expansion Mach wave