2.29 Numerical Fluid Mechanics
Spring 2015 – Lecture 13

REVIEW Lecture 12:

• Grid-Refinement and Error estimation
  – Estimation of the order of convergence and of the discretization error
  – Richardson’s extrapolation and Iterative improvements using Roomberg’s algorithm

• Fourier Error Analysis
  – Provide additional information to truncation error: indicates how well Fourier mode solution, i.e. wavenumber and phase speed, is represented
    • Effective wavenumber: \( \left( \frac{\partial e^{ikx}}{\partial x} \right)_j = i k_{\text{eff}} e^{ikx} \) (for CDS, 2\(^{nd}\) order, \( k_{\text{eff}} = \frac{\sin(k\Delta x)}{\Delta x} \))
    • Effective wave speed (for linear convection eqn., \( \frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = 0 \), integrating in time exactly):
      \[
      \frac{df_{k_{\text{num}}}}{dt} = -f_{k_{\text{num}}}(t) c i k_{\text{eff}} \Rightarrow f_{\text{numerical}}(x,t) = \sum_{k=-\infty}^{\infty} f_k(0) e^{ik(x-ct)} \Rightarrow \frac{c_{\text{eff}}}{c} = \frac{\sigma_{\text{eff}}}{\sigma} = \frac{k_{\text{eff}}}{k} \]
      (with \( \sigma_{\text{eff}} = -i k_{\text{eff}} c = -i k c_{\text{eff}} \))
REVIEW Lecture 12, Cont’d:

• Stability
  – Heuristic Method: trial and error
  – Energy Method: Find a quantity, \( l_2 \) norm \( \sum_j (\phi_j^n)^2 \), and then aim to show that it remains bounded for all \( n \).
    • Example: for \( \frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial x} = 0 \) we obtained \( 0 \leq \frac{c \Delta t}{\Delta x} \leq 1 \)
  – Von Neumann Method (Introduction), also called Fourier Analysis Method/Stability

• Hyperbolic PDEs and Stability
  – 2\textsuperscript{nd} order wave equation and waves on a string
    • Characteristic finite-difference solution (review)
    • Stability of C – C (CDS in time/space, explicit): \( C = \frac{c \Delta t}{\Delta x} < 1 \)
    • Example: Effective numerical wave numbers and dispersion
FINITE DIFFERENCES – Outline for Today

• Fourier Analysis and Error Analysis
  – Differentiation, definition and smoothness of solution for ≠ order of spatial operators

• Stability
  – Heuristic Method
  – Energy Method
  – Von Neumann Method (Introduction) : 1st order linear convection/wave eqn.

• Hyperbolic PDEs and Stability
  – Example: 2nd order wave equation and waves on a string
    • Effective numerical wave numbers and dispersion
  – CFL condition:
    • Definition
    • Examples: 1st order linear convection/wave eqn., 2nd order wave eqn., other FD schemes
  – Von Neumann examples: 1st order linear convection/wave eqn.
  – Tables of schemes for 1st order linear convection/wave eqn.

• Elliptic PDEs
  – FD schemes for 2D problems (Laplace, Poisson and Helmholtz eqns.)
References and Reading Assignments

- Lapidus and Pinder, 1982: Numerical solutions of PDEs in Science and Engineering. Section 4.5 on “Stability”.


Von Neumann Stability

- Widely used procedure
- Assumes initial error can be represented as a Fourier Series and considers growth or decay of these errors
- In theoretical sense, applies only to periodic BC problems and to linear problems
  - Superposition of Fourier modes can then be used
- Again, use, \( f(x,t) = \sum_{k=-\infty}^{\infty} f_k(t) e^{ikx} \) but for the error: \( \varepsilon(x,t) = \sum_{\beta=-\infty}^{\infty} \varepsilon_{\beta}(t) e^{i\beta x} \)
- Being interested in error growth/decay, consider only one mode: \( \varepsilon_{\beta}(t) e^{i\beta x} \approx e^{\gamma t} e^{i\beta x} \) where \( \gamma \) is in general complex and function of \( \beta \): \( \gamma = \gamma(\beta) \)
- Strict Stability: The error will not to grow in time if \( |e^{\gamma t}| \leq 1 \quad \forall \gamma \)
  - in other words, for \( t = n \Delta t \), the condition for strict stability can be written: \( |e^{\gamma \Delta t}| \leq 1 \) or for \( \xi = e^{\gamma \Delta t}, \quad |\xi| \leq 1 \) von Neumann condition

Norm of amplification factor \( \xi \) smaller or equal to 1

\[ f(x,t) = \sum_{k=-\infty}^{\infty} f_k(t) e^{ikx} \]
\[ \varepsilon(x,t) = \sum_{\beta=-\infty}^{\infty} \varepsilon_{\beta}(t) e^{i\beta x} \]

\[ |e^{\gamma t}| \leq 1 \quad \forall \gamma \]

\[ |e^{\gamma \Delta t}| \leq 1 \] or for \( \xi = e^{\gamma \Delta t}, \quad |\xi| \leq 1 \) von Neumann condition
Evaluation of the Stability of a FD Scheme

**Von Neumann Example**

- Consider again:
  \[
  \frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial x} = 0
  \]
- A possible FD formula ("upwind" scheme)
  \[
  \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c \frac{\phi_{j}^{n} - \phi_{j-1}^{n}}{\Delta x} = 0
  \]
  \((t = n\Delta t, \ x = j\Delta x)\) which can be rewritten:
  \[
  \phi_j^{n+1} = (1 - \mu) \phi_j^n + \mu \phi_{j-1}^n \quad \text{with} \quad \mu = \frac{c \Delta t}{\Delta x}
  \]
- Consider the Fourier error decomposition (one mode) and discretize it:
  \[
  \varepsilon(x,t) = \varepsilon_\beta(t) e^{i\beta x} = e^{\gamma t} e^{i\beta x} \Rightarrow \varepsilon_j^n = e^{\gamma n\Delta t} e^{i\beta j\Delta x}
  \]
- Insert it in the FD scheme, assuming the error mode satisfies the FD (strictly valid for linear eq. only):
  \[
  \varepsilon_{j}^{n+1} = (1 - \mu) \varepsilon_{j}^{n} + \mu \varepsilon_{j-1}^{n} \quad \Rightarrow \quad e^{\gamma (n+1)\Delta t} e^{i\beta j\Delta x} = (1 - \mu) e^{\gamma n\Delta t} e^{i\beta j\Delta x} + \mu e^{\gamma n\Delta t} e^{i\beta (j-1)\Delta x}
  \]
- Cancel the common term (which is \(\varepsilon_j^n = e^{\gamma n\Delta t} e^{i\beta j\Delta x}\)) in (linear) eq. and obtain:
  \[
  e^{\gamma \Delta t} = (1 - \mu) + \mu e^{-i\beta \Delta x}
  \]
Evaluation of the Stability of a FD Scheme
von Neumann Example

• The magnitude of $\xi = e^{i\Delta t}$ is then obtained by multiplying $\xi$ with its complex conjugate:

$$|\xi|^2 = \left((1 - \mu) + \mu e^{-i\beta\Delta x}\right)\left((1 - \mu) + \mu e^{i\beta\Delta x}\right) = 1 - 2\mu(1 - \mu)\left(1 - \frac{e^{i\beta\Delta x} + e^{-i\beta\Delta x}}{2}\right)$$

Since $\frac{e^{i\beta\Delta x} + e^{-i\beta\Delta x}}{2} = \cos(\beta\Delta x)$ and $1 - \cos(\beta\Delta x) = 2\sin^2\left(\frac{\beta\Delta x}{2}\right)$ \Rightarrow

$$|\xi|^2 = 1 - 2\mu(1 - \mu)\left(1 - \cos(\beta\Delta x)\right) = 1 - 4\mu(1 - \mu)\sin^2\left(\frac{\beta\Delta x}{2}\right)$$

• Thus, the strict von Neumann stability criterion gives

$$|\xi| \leq 1 \iff 1 - 4\mu(1 - \mu)\sin^2\left(\frac{\beta\Delta x}{2}\right) \leq 1$$

Since $\sin^2\left(\frac{\beta\Delta x}{2}\right) \geq 0 \ \forall \beta \ \ (1 - \cos(\beta\Delta x)) \geq 0 \ \forall \beta$ we obtain the same result as for the energy method:

$$|\xi| \leq 1 \iff \mu(1 - \mu) \geq 0 \iff 0 \leq \frac{c\Delta t}{\Delta x} \leq 1 \quad (\mu = \frac{c\Delta t}{\Delta x})$$

Equivalent to the CFL condition
Partial Differential Equations

Hyperbolic PDE: \[ B^2 - 4AC > 0 \]

Examples:

1. \[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]
   Wave equation, 2\(^{nd}\) order

2. \[ \frac{\partial u}{\partial t} \pm c \frac{\partial u}{\partial x} = 0 \]
   Sommerfeld Wave/radiation equation, 1\(^{st}\) order

3. \[ \frac{\partial u}{\partial t} + (U \cdot \nabla) u = g \]
   Unsteady (linearized) inviscid convection (Wave equation first order)

4. \[ (U \cdot \nabla) u = g \]
   Steady (linearized) inviscid convection

- Allows non-smooth solutions
- Information travels along characteristics, e.g.:
  - For (3) above: \[ \frac{dx_c}{dt} = U(x_c(t)) \]
  - For (4), along streamlines: \[ \frac{dx_c}{ds} = U \]
- Domain of dependence of \( u(x, T) \) = “characteristic path”
  - e.g., for (3), it is: \( x_c(t) \) for \( 0 < t < T \)
- Finite Differences, Finite Volumes and Finite Elements

- Upwind schemes
Waves on a String

\[ \frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad 0 < x < L, \quad 0 < t < \infty \]

Initial Conditions

\[ u(x,0) = f(x), \quad 0 \leq x \leq L \]

\[ u_t(x,0) = g(x), \quad 0 < x < L \]

Boundary Conditions

\[ u(0,t) = 0, \quad 0 < t < \infty \]

\[ u(L,t) = 0, \quad 0 < t < \infty \]

Wave Solutions

\[
\begin{align*}
  u &= \begin{cases} 
    F(x - ct) & \text{Forward propagating wave} \\
    G(x + ct) & \text{Backward propagating wave}
  \end{cases}
\end{align*}
\]

Typically Initial Value Problems in Time, Boundary Value Problems in Space

Time-Marching Solutions:

Implicit schemes generally stable
Explicit sometimes stable under certain conditions
Partial Differential Equations

Hyperbolic PDE - Example

Wave Equation

\[ \frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad 0 < x < L, \quad 0 < t < \infty \]

Discretization:

\[ h = \frac{L}{n} \]

\[ k = \frac{T}{m} \]

\[ x_i = (i-1)h, \quad i = 2, \ldots, n - 1 \]

\[ t_j = (j-1)k, \quad j = 1, \ldots, m \]

Finite Difference Representations (centered)

\[ u_{tt}(x,t) = \frac{u(x_i, t_{j-1}) - 2u(x_i, t_j) + u(x_i, t_{j+1})}{k^2} + O(k^2) \]

\[ u_{xx}(x,t) = \frac{u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j)}{h^2} + O(h^2) \]

\[ u_{i,j} = u(x_i, t_j) \]

Finite Difference Representations

\[ \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \]
Partial Differential Equations
Hyperbolic PDE - Example

Introduce Dimensionless Wave Speed

\[ C = \frac{ck}{h} \]

Explicit Finite Difference Scheme

\[ u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = C^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \]

Stability Requirement: \[ C = \frac{ck}{h} < 1 \]

\[ C = \frac{c \Delta t}{\Delta x} < 1 \] Courant-Friedrichs-Lewy condition (CFL condition)

Physical wave speed must be smaller than the largest numerical wave speed, or,

Time-step must be less than the time for the wave to travel to adjacent grid points:

\[ c < \frac{\Delta x}{\Delta t} \text{ or } \Delta t < \frac{\Delta x}{c} \]
Partial Differential Equations
Hyperbolic PDE - Example
Start of Integration: Euler and Higher Order Starters

Given ICs:
\[ u(x,0) = f(x), \quad 0 \leq x \leq L \]
\[ u_t(x,0) = g(x), \quad 0 < x < L \]

1\textsuperscript{st} order Euler Starter
\[ u_{i,2} = u(x_i, k) \simeq u(x_i, 0) + k u_t(x_i, 0) = f(x_i) + kg(x_i) \]

But, second derivative in \( x \) at \( t = 0 \) is known from IC:
\[ u_{xx}(x, 0) = f'' \]

From Wave Equation
\[ u_{tt}(x_i, 0) = c^2 u_{xx}(x_i, 0) = c^2 f_{xx}(x_i) = c^2 \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} + O(h^2) \]

Higher order Taylor Expansion
\[ u(x, k) = u(x, 0) + k u_t(x, 0) + \frac{u_{tt}(x, 0) k^2}{2} + O(k^3) \]

Higher Order Self Starter
\[ u_{i,2} = u(x_i, k) = f_i + kg_i + \frac{c^2 k^2}{2h^2}(f_{i-1} - 2f_i + f_{i+1}) + O(h^2 k^2) + O(k^3) \]

General idea: use the PDE itself to get higher order integration
**Waves on a String**

\[ \frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad 0 < x < L, \quad 0 < t < \infty \]

L=10;
T=10;
c=1.5;
N=100;
h=L/N;
M=400;
k=T/M;
C=c*k/h;
Lf=0.5;
x=[0:h:L]';
t=[0:k:T];
%fx='exp(-0.5*(5-x).^2/0.5^2).*cos((x-5)*pi)';
gx='0';
%Zero first time derivative at t=0
f=inline(fx,'x');
g=inline(gx,'x');
n=length(x);
m=length(t);
u=zeros(n,m);
%  Second order starter
u(2:n-1,1)=f(x(2:n-1));
for i=2:n-1
  u(i,2) = (1-C^2)*u(i,1) + k*g(x(i)) + C^2*(u(i+1,1)+u(i-1,1))/2;
end
% CDS: Iteration in time (j) and space (i)
for j=2:m-1
  for i=2:n-1
    u(i,j+1)=(2-2*C^2)*u(i,j) + C^2*(u(i+1,j)+u(i-1,j)) - u(i,j-1);
  end
end

figure(1)
plot(x,f(x));
a=title('fx = f(x)');
set(a,'FontSize',16);
figure(2)
wave(u',x,t);
a=xlabel('x');
set(a,'FontSize',14);
a=ylabel('t');
set(a,'FontSize',14);
a=title('Waves on String');
set(a,'FontSize',16);
colormap;
Waves on a String, Longer simulation: Effects of dispersion and effective wavenumber/speed

```matlab
L=10;
T=10;
c=1.5;
N=100;
h=L/N;  \% Horizontal resolution (Dx)
M=400;
\% Test: increase duration of simulation, to see effect of
dispersion and effective wavenumber/speed (due to 2\textsuperscript{nd} order)
T=100;M=4000;
k=T/M; \% Time resolution (Dt)
C=c\cdot k/h \% Try case C>1, e.g. decrease Dx or increase Dt
Lf=0.5;
x=[0:h:L]';
t=[0:k:T];
\%fx=[\exp(-0.5*(\text{	exttt{num2str}}(L/2)) \cdot x)^2/(\text{	exttt{num2str}}(L_lf \cdot x)^2)];
\%gx='0';
f=\exp(-0.5*(5-x)^2/0.5^2) \cdot \cos((x-5)*\pi);
gx='0';
f=inline(fx,'x');
g=inline(gx,'x');

n=length(x);
m=length(t);
u=zeros(n,m);
\%Second order starter
u(2:n-1,1)=f(x(2:n-1));
for i=2:n-1
    u(i,2) = (1-C^2)*u(i,1) + k\cdot g(x(i)) + C^2*(u(i-1,1)+u(i+1,1))/2;
end
\%CDS: Iteration in time (j) and space (i)
for j=2:m-1
    for i=2:n-1
        u(i,j+1)=(2-2*C^2)*u(i,j) + C^2*(u(i+1,j)+u(i-1,j)) - u(i,j-1);
    end
end
```

`waveeq.m`

Waves on String
Wave Equation d’Alembert’s Solution

Wave Equation
\[ \frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad 0 < x < L, \quad 0 < t < \infty \]

Solution
\[ u(x,t) = F(x - ct) + G(x + ct) \quad , \quad 0 < x < L \]

Periodicity Properties
\[ F(-z) = -F(z) \]
\[ F(z + 2L) = F(z) \]
\[ G(-z) = -G(z) \]
\[ G(z + 2L) = G(z) \]

Proof
\[ u_{xx}(x,t) = F''(x - ct) + G''(x + ct) \]
\[ u_{tt}(x,t) = c^2 F''(x - ct) + c^2 G''(x + ct) \]
\[ = c^2 u_{xx}(x,t) \]
Hyperbolic PDE
Method of Characteristics

Explicit Finite Difference Scheme

\[ u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = C^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \]

\[ u_{i,j+1} = (2 - 2C^2)u_{i,j} + C^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}, \ i = 2, \ldots, n-1 \]

First 2 Rows known

\[ u_{i,1} = u(x_i, 0) \]
\[ u_{i,2} = u(x_i, k) \]

Characteristic Sampling

\[ k = h/c \Rightarrow C = 1 \]

Exact Discrete Solution

\[ u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \]
Hyperbolic PDE
Method of Characteristics

Let's proof the following FD scheme is an exact Discrete Solution

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$

D’Alembert’s Solution with $C=1$

$$x_i - ct_j = (i - 1)h - c(j - 1)k$$
$$= (i - 1)h - (j - 1)h$$
$$= (i - j)h$$

$$x_i + ct_j = (i - 1)h + c(j - 1)k$$
$$= (i - 1)h + (j - 1)h$$
$$= (i + j - 2)h$$

$$u_{i,j} = F((i - j)h) + G((i + j - 2)h)$$

Proof

$$u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$
$$= F((i + 1 - j)h) + F((i - 1 - j)h) - F((i - (j - 1))h)$$
$$+ G((i + 1 + j - 2)h) + G((i - 1 + j - 2)h) - G((i + j - 1 - 2)h)$$
$$= F((i - (j + 1))h) + G((i + (j + 1) - 2)h)$$
$$= u_{i,j+1}$$
Courant-Fredrichs-Lewy Condition (1920’s)

• Basic idea: the solution of the Finite-Difference (FD) equation can not be independent of the (past) information that determines the solution of the corresponding PDE

• In other words:
  The “Numerical domain of dependence of FD scheme must include the mathematical domain of dependence of the corresponding PDE”

CFL NOT satisfied

CFL satisfied

PDE dependence

Numerical “stencil”
CFL: Linear convection (Sommerfeld Eqn) Example

Determine domain of dependence of PDE and of FD scheme

- **PDE:** \[ \frac{\partial u(x,t)}{\partial t} + c \frac{\partial u(x,t)}{\partial x} = 0 \] Characteristics: If \( \frac{dx}{dt} = c \) \( \Rightarrow \) \( x = c \, t + \zeta \) and \( du = 0 \) \( \Rightarrow u = \text{cst} \)

Solution of the form: \( u(x,t) = F(x - ct) \)

- **FD scheme.** For our Upwind discretization, with \( t = n \Delta t, \, x = j \Delta x \) :

\[
\frac{\phi_{j}^{n+1} - \phi_{j}^{n}}{\Delta t} + c \frac{\phi_{j}^{n} - \phi_{j-1}^{n}}{\Delta x} = 0
\]

Slope of characteristic: \( \frac{dt}{dx} = \frac{1}{c} \)

Slope of Upwind scheme: \( \frac{\Delta t}{\Delta x} \)

\( \Rightarrow \) CFL condition: \( \frac{\Delta t}{\Delta x} \leq \frac{1}{c} \)

\[ \frac{c \, \Delta t}{\Delta x} \leq 1 \]

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Source: Figure 2.1 from Durran, D. Numerical Methods for Wave Equations in Geophysical Fluid Dynamics. Springer, 1998.
CFL: 2nd order Wave equation Example

Determine domain of dependence of PDE and of FD scheme

- PDE, second order wave eqn example:

\[ \frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad 0 < x < L, \quad 0 < t < \infty \]

- As seen before: \( u(x,t) = F(x - ct) + G(x + ct) \) \( \Rightarrow \) slope of characteristics: \( \frac{dt}{dx} = \pm \frac{1}{c} \)

- FD scheme: discretize: \( t = n\Delta t, \quad x = j\Delta x \)

- CD scheme (CDS) in time and space (2nd order), explicit

\[ \frac{u_{j}^{n+1} - 2u_{j}^{n} + u_{j}^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^{n} - 2u_{j}^{n} + u_{j-1}^{n-1}}{\Delta x^2} \quad \Rightarrow \quad u_{j}^{n+1} = (2 - 2C^2)u_{j}^{n} + C^2(u_{j+1}^{n} + u_{j-1}^{n}) - u_{j}^{n-1} \quad \text{where} \quad C = \frac{c\Delta t}{\Delta x} \]

- We obtain from the respective slopes:

\[ \frac{c \Delta t}{\Delta x} \leq 1 \]

Full line case: CFL satisfied

Dotted lines case:
\( c \) and \( \Delta t \) too big, \( \Delta x \) too small (CFL NOT satisfied)
CFL Condition: Some comments

- CFL is only a necessary condition for stability
- Other (sufficient) stability conditions are often more restrictive
  - For example: if \( \frac{\partial u(x,t)}{\partial t} + c \frac{\partial u(x,t)}{\partial x} = 0 \) is discretized as
  
  \[
  \left( \frac{\partial u(x,t)}{\partial t} \right)_{CD,2^{nd} \text{order in } t} + c \left( \frac{\partial u(x,t)}{\partial x} \right)_{CD,4^{th} \text{order in } x} \approx 0
  \]

  - One obtains from the CFL: \( \frac{c \Delta t}{\Delta x} \leq 2 \)

  - While a Von Neumann analysis leads: \( \frac{c \Delta t}{\Delta x} \leq 0.728 \)

- For equations that are not purely hyperbolic or that can change of type (e.g. as diffusion term increases), CFL condition can at times be violated locally for a short time, without leading to global instability further in time
von Neumann Examples

• Forward in time (Euler), centered in space, Explicit

\[ \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c \frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x} = 0 \quad \Rightarrow \quad \phi_j^{n+1} = \phi_j^n - \frac{C}{2} (\phi_{j+1}^n - \phi_{j-1}^n) \]

– Von Neumann: insert \( \varepsilon(x,t) = \varepsilon_{\beta}(t) e^{i\beta x} = e^{\gamma t} e^{i\beta x} \) \( \Rightarrow \) \( \varepsilon_j^n = e^{\gamma n\Delta t} e^{i\beta j\Delta x} \)

\[ \Rightarrow \varepsilon_{j+1}^n = \varepsilon_j^n - \frac{C}{2} (\varepsilon_{j+1}^n - \varepsilon_{j-1}^n) \quad \Rightarrow \quad e^{\gamma\Delta t} = 1 - \frac{C}{2} (e^{i\beta\Delta x} - e^{-i\beta\Delta x}) = 1 - Ci \sin(\beta\Delta x) \]

• Taking the norm:

\[ |e^{\gamma t}|^2 = |\xi|^2 = (1 - Ci \sin(\beta\Delta x))(1 + Ci \sin(\beta\Delta x)) = 1 + C^2 \sin^2(\beta\Delta x) \quad \geq 1 \text{ for } C \neq 0 ! \]

• Unconditionally Unstable

• Implicit scheme (backward in time, centered in space)

\[ \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c \frac{\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}}{2\Delta x} = 0 \quad \Rightarrow \quad \varepsilon_{j+1}^{n+1} = \varepsilon_j^n - \frac{C}{2} (\varepsilon_{j+1}^{n+1} - \varepsilon_{j-1}^{n+1}) \quad \Rightarrow \quad e^{\gamma\Delta t} = 1 - e^{\gamma\Delta t} Ci \sin(\beta\Delta x) \]

\[ \Rightarrow \quad |e^{\gamma t}|^2 = |\xi|^2 = \frac{1}{(1 + Ci \sin(\beta\Delta x))(1 - Ci \sin(\beta\Delta x))} = \frac{1}{1 + C^2 \sin^2(\beta\Delta x)} \quad \leq 1 \text{ for } C \neq 0 ! \]

• Unconditionally Stable
Stability of FD schemes for $u_t + b u_y = 0$  

(t denoted x below)

Stability of FD schemes for \( u_t + b \, u_y = 0 \), Cont.

Partial Differential Equations
Elliptic PDE

Laplace Operator
\[ \nabla^2 \equiv u_{xx} + u_{yy} \]

Examples:

- Laplace Equation – Potential Flow
  \[ \nabla^2 \phi = 0 \]
- Poisson Equation
  \[ \nabla^2 \phi = g(x, y) \]
- Helmholtz equation – Vibration of plates
  \[ \nabla^2 u + f(x, y)u = 0 \]
- Steady Convection-Diffusion
  \[ \mathbf{U} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u} \]

- Smooth solutions ("diffusion effect")
- Very often, steady state problems
- Domain of dependence of \( u \) is the full domain \( \mathbf{D}(x,y) \) => “global” solutions
- Finite differ./volumes/elements, boundary integral methods (Panel methods)
Partial Differential Equations
Elliptic PDE - Example

0 ≤ x ≤ a , 0 ≤ y ≤ b;

Equidistant Sampling
\[ h = \frac{a}{(n - 1)} \]
\[ h = \frac{b}{(m - 1)} \]

Discretization
\[ x_i = (i - 1)h, \; i = 1, \ldots, n \]
\[ y_j = (j - 1)h, \; j = 1, \ldots, m \]

Finite Differences
\[ u_{xx}(x, t) = \frac{u(x_{i-1}, y_j) - 2u(x_i, y_j) + u(x_{i+1}, y_j)}{h^2} + O(h^2) \]
\[ u_{yy}(x, t) = \frac{u(x_i, y_{j-1}) - 2u(x_i, y_j) + u(x_i, y_{j+1})}{h^2} + O(h^2) \]

Dirichlet BC
Partial Differential Equations
Elliptic PDE - Example

Discretized Laplace Equation

\[ \nabla^2 u = \frac{u(x_{i-1}, y_j) + u(x_i, y_{j-1}) - 4u(x_i, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j+1})}{h^2} = 0 \]

\[ u_{i,j} = u(x_i, t_j) \]

Finite Difference Scheme

\[ u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 0 \]

Boundary Conditions

\[ u(x_1, y_j) = u_{1,j}, \quad 2 \leq j \leq m - 1 \]
\[ u(x_n, y_j) = u_{n,j}, \quad 2 \leq j \leq m - 1 \]
\[ u(x_i, y_1) = u_{i,1}, \quad 2 \leq i \leq n - 1 \]
\[ u(x_i, y_n) = u_{i,n}, \quad 2 \leq i \leq n - 1 \]

Global Solution Required

\[ u(x, b) = f_2(x) \]
\[ u(0, y) = g_1(y) \]
\[ u(a, y) = g_2(y) \]
\[ u(x, 0) = f_1(x) \]
Elliptic PDEs
Laplace Equation, Global Solvers

Dirichlet BC

Leads to $Ax = b$, with $A$ block-tridiagonal:

$$A = \text{tri} \{ I, T, I \}$$

\[
\begin{align*}
-p_1 + p_2 + p_4 &= -u_{2,1} - u_{1,2} \\
p_1 - 4p_2 + p_3 + p_6 &= -u_{3,1} \\
p_2 - 4p_3 + p_5 &= -u_{4,1} - u_{5,2} \\
-p_4 + p_5 + p_7 &= -u_{1,3} \\
p_4 - 4p_5 + p_6 + p_8 &= 0 \\
p_5 - 4p_6 + p_7 + p_9 &= -u_{5,3} \\
p_5 - 4p_7 + p_8 &= -u_{2,5} - u_{1,4} \\
p_7 - 4p_8 + p_9 &= -u_{3,5} \\
p_5 + p_8 - 4p_9 &= -u_{4,5} - u_{5,4}
\end{align*}
\]
Elliptic PDEs

Neumann Boundary Conditions

Neumann (Derivative) Boundary Condition

\[ \frac{\partial}{\partial N} u(x, y) \quad \text{given} \]

Finite Difference Scheme at \( i = n \)

\[ u_{n+1,j} + u_{n-1,j} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j} = 0 \]

Derivative BC: Finite Difference

\[ \frac{u_{n+1,j} - u_{n-1,j}}{2h} \approx u_x(x_n, y_j) \]

Boundary Finite Difference Scheme at \( i = n \)

\[ u_{n-1,j} + 2\Delta x \frac{\partial u}{\partial x} \bigg|_{x_n} + u_{n-1,j} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j} = 0 \]

Leads to a factor 2 (a matrix 2 \( \mathbf{I} \) in \( \mathbf{A} \)) for points along boundary

Finite Difference Scheme

\[ u(x, b) = f_2(x) \]

\[ u(0, y) = g_1(y) \]

\[ u_x(a, y) \text{ given} \]
2.29 Numerical Fluid Mechanics
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