The 3D wave equation

In three-dimensions, the Wave Equation is generalized as

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0.$$ 

Our familiar plane and spherical waves are special solutions.
Planar and Spherical Wavefronts

Planar wavefront (plane wave):

The wave phase is constant along a planar surface (the wavefront).

As time evolves, the wavefronts propagate at the wave speed without changing; we say that the wavefronts are invariant to propagation in this case.

Spherical wavefront (spherical wave):

The wave phase is constant along a spherical surface (the wavefront).

As time evolves, the wavefronts propagate at the wave speed and expand outwards while preserving the wave’s energy.
Wavefronts, rays, and wave vectors

Rays are:
1) normals to the wavefront surfaces
2) trajectories of “particles of light”

Wave vectors:
At each point on the wavefront, we may assign a normal vector \( \mathbf{k} \)

This is known as the wave vector; it magnitude \( k \) is the wave number and it is defined as

\[
k \equiv |\mathbf{k}| = \frac{2\pi}{\lambda}
\]
3D wave vector from the wave equation

We try a sinusoidal solution

\[ a \exp \{ i (k_x x + k_y y + k_z z - \omega t) \} = \]

\[ = a \exp \{ i (\mathbf{k} \cdot \mathbf{r} - \omega t) \}, \quad \text{where} \]

\[ \mathbf{k} = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z \quad \text{is the wave vector, and} \]

\[ \mathbf{r} = \hat{x} x + \hat{y} y + \hat{z} z \quad \text{is the Cartesian} \]

coordinate vector, to the 3D wave equation

\[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0 \Rightarrow \]

\[ -a \left( k_x^2 + k_y^2 + k_z^2 - \frac{\omega^2}{c^2} \right) e^{i(k_x x + k_y y + k_z z - \omega t)} = 0 \Rightarrow \]

\[ k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}. \]

That is, \[ |\mathbf{k}| = \frac{\omega}{c} = \frac{2\pi n}{\lambda} \equiv k \quad \text{(wave number.)} \]

The wavefront is the surface \[ \mathbf{k} \cdot \mathbf{r} = \text{const.} \]

\[ i.e., \quad \text{the locus of points on the wave} \]

that have the same phase (modulo \( 2\pi \))

after propagating by the same time \( t \).
3D wave vector and the Descartes sphere

The wave vector represents the momentum of the wave. Consistent with Geometrical Optics, its magnitude is constrained to be proportional to the refractive index $n$ ($2\pi/\lambda_{\text{free}}$ is a normalization factor).

$$k \equiv |\mathbf{k}| = \frac{2\pi n}{\lambda_{\text{free}}}$$

In wave optics, the Descartes sphere is also known as Ewald sphere or simply as the $k$-sphere. (Ewald sphere may be familiar to you from solid state physics.)
The wavefront in this case is a sphere

\[ kr = \text{const.}, \quad \text{where} \quad r \equiv |\mathbf{r}|. \]

Without proof (pls. see the textbook) we assert

\[ f(\mathbf{r}, t) = a \frac{\cos(kr - \omega t - \pi/2)}{r}, \]

In complex representation,

\[ \hat{f}(\mathbf{r}, t) = a \frac{\exp\{i (kr - \omega t)\}}{i r}, \]

and in phasor notation (dropping the \( e^{-i\omega t} \))

\[ \hat{f}(\mathbf{r}) = \frac{a}{i r} \exp\{ikr\}. \]

In the paraxial approximation, \( z \gg |x|, |y| \) so

\[ r = \sqrt{x^2 + y^2 + z^2} = z \sqrt{1 + \frac{x^2 + y^2}{z^2}} \approx z + \frac{x^2 + y^2}{2z} \Rightarrow \]

\[ \hat{f}(\mathbf{r}) \approx \frac{a}{i z} \exp\{ikz\} \exp\left\{ik \frac{x^2 + y^2}{2z}\right\} \]

\[ = \frac{a}{i z} \exp\left\{i \frac{2\pi}{\lambda} z\right\} \exp\left\{i \frac{\pi}{\lambda z} x^2 + y^2\right\}. \]
Dispersive waves

We have learnt from Geometrical Optics that the speed of light can be wavelength dependent, e.g. due to material dispersion \( n(\lambda) \). This means that the wave equation for light waves must be written as

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{c^2(k)} \frac{\partial^2 f}{\partial t^2},
\]

where \( c(k) = c_{\text{vacuum}}/n(k) \) denotes the dependence of \( c \) on the wave number \( k = 2\pi/\lambda \). This kind of wave is called dispersive.

Another example of a dispersive wave is a guided wave. It turns out that, due to the boundary conditions at the waveguide’s edge, the simple dispersion relation \( c = \lambda\nu \) does not hold for a waveguide, and it must be replaced by a different relationship. Without going into the details here, the dispersion relationship for the metal waveguide shown on the left is

\[
\left( \frac{m\pi}{a} \right)^2 + k^2 = \left( \frac{\omega}{c} \right)^2, \quad m = 0, \pm 1, \pm 2, \ldots
\]
Consider two waves of different frequency and wavelength
\[ f_1(z, t) = a \cos (k_1 z - \omega_1 t), \quad f_2(z, t) = a \cos (k_2 z - \omega_2 t). \]

Their superposition is
\[
f(z, t) = f_1(z, t) + f_2(z, t) = a \cos (k_1 z - \omega_1 t) + a \cos (k_2 z - \omega_2 t) = 2a \cos \left( \frac{k_1 + k_2}{2} z - \frac{\omega_1 + \omega_2}{2} t \right) \cos \left( \frac{k_1 - k_2}{2} z - \frac{\omega_1 - \omega_2}{2} t \right) = 2a \cos \left( k_c z - \omega_c t \right) \cos \left( k_m z - \omega_m t \right),
\]

where \( k_c \equiv \frac{k_1 + k_2}{2} \) and \( \omega_c \equiv \frac{\omega_1 + \omega_2}{2}; \)
are the wave vector and frequency of the carrier wave; and \( k_m \equiv \frac{k_1 - k_2}{2} \) and \( \omega_m \equiv \frac{\omega_1 - \omega_2}{2}; \)
are the wave vector and frequency of the modulation. \( \omega_m \)
is also referred to as beat frequency.

If \( \omega_1 \approx \omega_2 \) and \( k_1 \approx k_2, \) then
\[
v_p = \frac{\omega}{k}, \quad \text{and} \quad v_g = \frac{\partial \omega}{\partial k} \bigg|_{\omega = \omega(k)}.
\]
The carrier wave propagates at the phase velocity \( v_p \equiv \frac{\omega_c}{k_c}, \)
whereas the modulation propagates at the group velocity \( v_g \equiv \frac{\omega_m}{k_m}. \)
Group and phase velocity

\[ v_p = \frac{\omega}{k} \bigg|_{\omega=\omega(k_c)} \]

\[ v_g = \frac{\partial \omega}{\partial k} \bigg|_{\omega=\omega(k_c)} \]

Note \( v_g < c < v_p \)
Spatial frequencies

We now turn to a monochromatic (single color) optical field. The field is often observed (or measured) at a planar surface along the optical axis $z$. The wavefront shape at the observation plane is, therefore, of particular interest.
Spatial frequencies

\[ E(x, z) = \exp \left\{ \frac{2\pi}{\lambda} (x \sin \theta + z \cos \theta) \right\} \]
\[ = \exp \left\{ i2\pi \left( \frac{x}{\Lambda} + \text{const} \right) \right\} \]

\[ \Lambda \equiv \frac{\lambda}{\sin \theta} \]

\[ z = \text{fixed} \]

Plane wave

\[ E(x, z) \sim \exp \left\{ \frac{2\pi}{\lambda} \sqrt{x^2 + y^2 + z^2} \right\} \]
\[ \approx \exp \left\{ i\frac{2\pi z}{\lambda} \right\} \exp \left\{ i\pi \frac{x^2 + y^2}{\lambda z} \right\} \]

Spatial frequency

Chirped frequency \[ f_x (x) \sim \frac{2x}{\lambda z} \]

Spherical wave

\( x \)

\( x \)

\( z = \text{fixed} \)

\( t=10.0 \)

\( t=10.0 \)
Today

• Electromagnetics
  – Electric (Coulomb) and magnetic forces
  – Gauss Law: electrical
  – Gauss Law: magnetic
  – Faraday’s Law
  – Ampère-Maxwell Law
  – Maxwell’s equations ⇒ Wave equation
  – Energy propagation
    • Poynting vector
    • average Poynting vector: intensity
  – Calculation of the intensity from phasors
    • Intensity
Electric and magnetic forces

Coulomb force

\[ |\mathbf{F}| = \frac{1}{4\pi\varepsilon_0} \frac{qq'}{r^2} \]

(dielectric) permitivity of free space

\[ \varepsilon_0 = 8.8542 \times 10^{-12} \, \text{Cb}^2/\text{N} \cdot \text{m}^2 \]

Magnetic force

\[ \frac{d|\mathbf{F}|}{dl} = \mu_0 \frac{II'}{2\pi r} \]

(magnetic) permeability of free space

\[ \mu_0 = 4\pi \times 10^{-7} \, \text{N} \cdot \text{sec}^2/\text{Cb}^2 \]
Electric and magnetic fields

**Observation**

- Electric field $E$
- Magnetic induction $B$
- Electric charge $q$
- Velocity $v$

**Generation**

- Static charge: $\Rightarrow$ Electric field $E$
- Moving charge (electric current): $\Rightarrow$ Magnetic field $B$

Lorentz force

$$F = q(E + v \times B)$$
Gauss Law: electric field

\[ \oint_A \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} \iiint_V \rho dV \]

\[ \mathbf{E} \cdot d\mathbf{a} \]

\[ \rho \text{: charge density} \]

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]
Gauss Law: magnetic field

\[ \oint_A \mathbf{B} \cdot d\mathbf{a} = 0 \]

Gauss theorem \[ \nabla \cdot \mathbf{B} = 0 \]
Faraday’s Law: electromotive force

\[ \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_A \mathbf{B} \cdot d\mathbf{a} \]

Stokes theorem

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]
Ampère-Maxwell Law: magnetic induction

\[ \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left( \iint_A \mathbf{J} \cdot d\mathbf{a} + \varepsilon_0 \iint_A \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} \right) \quad \text{Stokes theorem} \]

\[ \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \]
Maxwell’s Equations (free space)

Integral form

\[ \iint_A \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} \iiint_V \rho dV \]

\[ \iint_A \mathbf{B} \cdot d\mathbf{a} = 0 \]

\[ \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_A \mathbf{B} \cdot d\mathbf{a} \]

\[ \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left( \iint_A \mathbf{J} \cdot d\mathbf{a} + \varepsilon_0 \iint_A \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} \right) \]

Differential form

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

\[ \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \]
Wave Equation for electromagnetic waves

We will derive the Wave Equation from Maxwell’s electromagnetic equations in free space and in the absence of charges and currents. Starting from Faraday’s equation,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t}$$

Now we substitute Ampère–Maxwell’s Law

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

the following identity from vector calculus

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E},$$

and Gauss’ Law for electric fields,

$$\nabla \cdot \mathbf{E} = 0.$$

Collecting all these results, we obtain

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

Comparing with the 3D Wave Equation,

$$\nabla^2 f - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0,$$

we see that each component of the vector \( \mathbf{E} \) satisfies the Wave Equation with velocity

$$\frac{1}{c^2} = \mu_0 \epsilon_0 \Rightarrow c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Since \( \epsilon_0 = 8.8542 \times 10^{-12} \text{Cb}^2/\text{N} \cdot \text{m}^2 \), \( \mu_0 = 4\pi \times 10^{-7} \text{N} \cdot \text{sec}^2/\text{Cb}^2 \), we obtain the speed of electromagnetic waves in vacuum

$$c = 3 \times 10^8 \frac{\text{m}}{\text{sec}}.$$