Overview

• Last lecture:
  – Wavefronts and rays
  – Fermat’s principle
  – Reflection
  – Refraction

• Today: two applications of Fermat’s principle to the problem of *perfectly* focusing a plane wave to a point:
  – paraboloidal reflector
  – ellipsoidal refractor
Curved reflecting surfaces

radiation from far away source (at infinity)

reflective dish

detector

Applications:
solar concentrators, satellite dishes, radio telescopes

Image by hyperborea at Flickr.

Courtesy of NASA/JPL-Caltech.
Paraboloidal reflector: *perfect focusing*

What should the shape function $s(x)$ be in order for the incoming parallel ray bundle to come to perfect focus?

The easiest way to find the answer is to invoke Fermat’s principle: since the rays from infinity follow the *minimum* path before they meet at $P$, it follows that they must follow the *same* path.

$$2f = f - s + \sqrt{x^2 + (f - s)^2}$$

$$f + s = \sqrt{x^2 + (f - s)^2}$$

$$x^2 = (f + s)^2 - (f - s)^2 \Rightarrow$$

$$= 4sf \Rightarrow$$

$$s(x) = \frac{x^2}{4f}$$

A paraboloidal reflector focuses a normally incident plane wave to a point.
Ellipsoidal refractor: *perfect focusing*

What should the shape function $s(x)$ be in order for the incoming parallel ray bundle to come to perfect focus?

Once again, we invoke Fermat’s principle: since the rays from infinity follow the *minimum* path before they meet at $P$, it follows that they must follow the *same* path.

$$nf = s + n\sqrt{x^2 + (f-s)^2}$$

$$\Rightarrow \ldots \Rightarrow$$

$$(n^2 - 1)s^2 - 2n(n-1)fs + n^2x^2 = 0$$

$$\Rightarrow \ldots \Rightarrow$$

$$\left(s - \frac{n}{n+1}f\right)^2 + \frac{n^2}{n^2-1}x^2 = \left(\frac{n}{n+1}f\right)^2$$

A ellipsoidal refractor focuses a normally incident plane wave to a point.
Ellipsoidal refractive concentrator

Surface shape $s(x)$:

$$
\left( s - \frac{n}{n+1}f \right)^2 + \frac{n^2}{n^2 - 1} x^2 = \left( \frac{n}{n+1}f \right)^2
$$
Overview

• Last lecture:
  • focusing ray bundles coming from infinity (plane waves)
    – paraboloidal reflector
    – ellipsoidal refractor
• Today:
  – spherical and plane waves
  – perfect focusing and collimation elements:
    • paraboloidal mirrors, ellipsoidal and hyperboloidal refractors
  – imperfect focusing: spherical elements
  – the paraxial approximation
  – ray transfer matrices
• Next lecture:
  – ray tracing using the matrix approach
Spherical and plane waves

diverging spherical wave

converging spherical wave

**spherical wave-fronts**
(by definition \( \perp \) to divergent fan of rays)

spherical wave at infinity \( \Leftrightarrow \) plane wave

**planar wave-fronts**
(by definition \( \perp \) to parallel fan of rays)

point image
Perfect imaging of point sources at infinity

\[ s = \frac{x^2}{4f} \]

Paraboloidal reflector

\[ \left( s - \frac{n}{n+1}f \right)^2 + \frac{n^2}{n^2-1}x^2 = \left( \frac{n}{n+1}f \right)^2 \]

Ellipsoidal refractor

focus at \( F \)
Perfect imaging of point sources to infinity

Paraboloidal reflector

Hyperboloidal refractor

\[ s = \frac{x^2}{4f} \]

\[ \left( s + \frac{1}{n+1}f \right)^2 - \frac{1}{n^2 - 1} x^2 = \left( \frac{1}{n+1}f \right)^2 \]
Summary: objects and images at infinity

Object at $\infty$ image at $F$

Paraboloidal reflector

$$s = \frac{x^2}{4f}$$

Ellipsoidal refractor

$$\left(s - \frac{n}{n+1}f\right)^2 + \frac{n^2}{n^2-1}x^2 = \left(\frac{n}{n+1}f\right)^2$$

Hyperboloidal refractor

$$\left(s - \frac{1}{n+1}f\right)^2 - \frac{1}{n^2-1}x^2 = \left(\frac{1}{n+1}f\right)^2$$

Object at $F$ image at $\infty$

Paraboloidal reflector

$$s = \frac{x^2}{4f}$$

Ellipsoidal refractor

$$\left(s + \frac{n}{n+1}f\right)^2 + \frac{n^2}{n^2-1}x^2 = \left(\frac{n}{n+1}f\right)^2$$

Hyperboloidal refractor

$$\left(s + \frac{1}{n+1}f\right)^2 - \frac{1}{n^2-1}x^2 = \left(\frac{1}{n+1}f\right)^2$$

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Focusing: from planar to spherical wavefronts

- The wavefronts are spaced by \( \lambda \) in air, by \( \lambda/n \) in the dielectric medium.
- The wavefronts remain continuous at the interface.
- Refraction at the curved interface causes the wavefronts to bend.
- The elliptical shape of the refractive interface at on-axis incidence works out exactly so the planar wavefronts become spherical inside the dielectric medium.
  \( \Rightarrow \) perfect focusing results (within the approximations of geometrical optics).
- Any shape other than elliptical or off-axis incidence would have resulted in a non-spherical wavefront, therefore imperfect focusing.
  - such imperfectly focusing wavefronts are called aberrated.
The need for “perfect imagers”

We can think of the job of the imaging system as “mapping” point sources emanating from scattering of the incident light by object points. Ideally, each object point should be mapped onto a single point image.

However, we saw that even the asphere cannot do a perfect imaging job except for object points on or near the axis. Therefore, imaging can be achieved only approximately.
Perfect imaging on-axis

The purpose of the simplest imaging system is to convert a diverging spherical wave to a converging spherical wave, i.e. to image a point object to a point image.

This ideal imaging element is referred to as asphere, (“not a sphere” in Greek) or as aspheric lens.

It works perfectly on axis and reasonably well in a limited range of angles off-axis. Manufacturing constraints usually limit refractive elements to spherical surfaces.
Aberrated imaging with spherical elements

If the asphere is replaced by a sphere, the refracted wavefront inside the sphere is not planar; neither is the refracted wavefront after the sphere spherical.

The resulting image is imperfect, or **aberrated**, because the converging rays fail to focus (intersect) at a single point.

Confusingly enough, this type of imperfect imaging is referred to as **spherical aberration**.

We will learn about more types of aberrations in detail later.
Refraction from a sphere: paraxial approximation

Snell’s Law at the interface: \( n_{\text{left}} \sin \theta_{\text{left}} = n_{\text{right}} \sin \theta_{\text{right}} \)

From the geometry: \( \theta_{\text{left}} = \alpha_{\text{left}} + \phi, \quad \theta_{\text{right}} = \alpha_{\text{right}} + \phi \)

\[ \Rightarrow n_{\text{left}} (\sin \alpha_{\text{left}} \cos \phi + \cos \alpha_{\text{left}} \sin \phi) = \]

\[ = n_{\text{right}} (\sin \alpha_{\text{right}} \cos \phi + \cos \alpha_{\text{right}} \sin \phi) \]

Now assume \( x \ll R, \alpha_{\text{left}} \ll 1, \alpha_{\text{right}} \ll 1. \)

This set of assumptions constitutes the paraxial approximation.

From this follows:

\[ \sin \alpha_{\text{left}} \approx \alpha_{\text{left}}; \quad \cos \alpha_{\text{left}} \approx 1; \quad \sin \alpha_{\text{right}} \approx \alpha_{\text{right}}; \quad \cos \alpha_{\text{right}} \approx 1. \]

\[ \sin \phi = \frac{x}{R}, \quad \cos \phi \approx 1. \]

\[ \Rightarrow n_{\text{left}} \left( \alpha_{\text{left}} + \frac{x}{R} \right) = n_{\text{right}} \left( \alpha_{\text{right}} + \frac{x}{R} \right) \Rightarrow n_{\text{right}} \alpha_{\text{right}} = n_{\text{left}} \alpha_{\text{left}} + \frac{n_{\text{left}} - n_{\text{right}}}{R} x \]
Free space propagation: paraxial approximation

Consider two positions, separated by distance $d$, along a ray propagating in free space of uniform index of refraction $n_{\text{left}} = n_{\text{right}} \equiv n$.

According to the Fermat principle, $n_{\text{left}} \alpha_{\text{left}} = n_{\text{right}} \alpha_{\text{right}}$.

From the geometry we find $x_{\text{right}} = x_{\text{left}} + d \tan \alpha_{\text{left}} \approx x_{\text{left}} + d \alpha_{\text{left}}$, since $\tan \alpha_{\text{left}} \approx \alpha_{\text{left}}$ in the paraxial approximation $\alpha_{\text{left}} \ll 1$. 
Ray transfer matrices

\[ n_{right} \alpha_{right} = n_{left} \alpha_{left} \]

\[ x_{right} = x_{left} + d \alpha_{left} \]

or, in matrix form:

\[
\begin{pmatrix}
  n_{right} \alpha_{right} \\
  x_{right}
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 \\
  d/n_{left} & 1
\end{pmatrix}
\begin{pmatrix}
  n_{left} \alpha_{left} \\
  x_{left}
\end{pmatrix} =
\begin{pmatrix}
  n_{right} \alpha_{right} \\
  x_{right}
\end{pmatrix} =
\begin{pmatrix}
  1 & -n_{right} - n_{left} \\
  0 & R
\end{pmatrix}
\begin{pmatrix}
  n_{left} \alpha_{left} \\
  x_{left}
\end{pmatrix}
\]