Experimentation and Robust Design of Engineering Systems


and


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Research Overview

Concept Design

Complex Systems

Adaptive Experimentation and Robust Design

Outreach to K-12

Methodology Validation

\[
\Pr \left( \beta \varepsilon \right) > 0 | \beta \varepsilon > \beta \varepsilon \right) = \frac{1}{\pi} \left( 2^{(n)} \right) \sqrt{\frac{1}{2 \sigma_{int}^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\frac{\left( x_1 - \mu \right)^2}{2 \sigma_{int}^2}} x_2 \left( \sigma_{int}^2 + (n-2)\sigma_{int}^2 + \frac{1}{2} \sigma^2 \right)}{\sigma_{int} \sqrt{\sigma_{int}^2 + (n-2)\sigma_{int}^2 + \frac{1}{2} \sigma^2}} dx_1 dx_2
\]
Outline

• Introduction
  – History
  – Motivation

• Recent research
  – Adaptive experimentation
  – Robust design
“An experiment is simply a question put to nature … The chief requirement is simplicity: only one question should be asked at a time.”

“To call in the statistician after the experiment is done may be no more than asking him to perform a post-mortem examination: he may be able to say what the experiment died of.”

Image removed due to copyright restrictions. Please see Fig. 1 in Fisher, R. A. “The Arrangement of Field Experiments.” *Journal of the Ministry of Agriculture of Great Britain* 33 (1926): 503-513.
Estimation of Factor Effects

Say the independent experimental error of observations \((a), (ab), \text{et cetera}\) is \(\sigma_\varepsilon\).

We define the main effect estimate \(A\) to be

\[
A \equiv \frac{1}{4} \left[ (abc) + (ab) + (ac) + (a) - (b) - (c) - (bc) - (1) \right]
\]

The standard deviation of the estimate is

\[
\sigma_A = \frac{1}{4} \sqrt{8\sigma_\varepsilon} = \frac{1}{2} \sqrt{2\sigma_\varepsilon}
\]

A factor of two improvement in efficiency as compared to “single question methods”
Fractional Factorial Experiments

“It will sometimes be advantageous deliberately to sacrifice all possibility of obtaining information on some points, these being confidently believed to be unimportant … These comparisons to be sacrificed will be deliberately confounded with certain elements of the soil heterogeneity… Some additional care should, however, be taken…”

Fractional Factorial Experiments

$2^{3-1}_{III}$
Fractional Factorial Experiments

<table>
<thead>
<tr>
<th>Trial</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
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27-4 Design (aka “orthogonal array”)  
Every factor is at each level an equal number of times (balance).  
High replication numbers provide precision in effect estimation.  
Resolution III.
Robust Parameter Design

Robust Parameter Design … is a statistical / engineering methodology that aims at reducing the performance variation of a system (i.e. a product or process) by choosing the setting of its control factors to make it less sensitive to noise variation.

Cross (or Product) Arrays


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<tr>
<th>Control Factors</th>
<th>A</th>
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2^{7-4}_{III} \times 2^{3-1}_{III}
Step 4 Summary:
- Determine control factor levels
- Calculate the DOF
- Determine if there are any interactions
- Select the appropriate orthogonal array
One way of thinking of the great advances of the science of experimentation in this century is as the final demise of the “one factor at a time” method, although it should be said that there are still organizations which have never heard of factorial experimentation and use up many man hours wandering a crooked path.

My Observations of Industry

• Farming equipment company has reliability problems
• Large blocks of robustness experiments had been planned at outset of the design work
• More than 50% were not finished
• Reasons given
  – Unforeseen changes
  – Resource pressure
  – Satisficing

“Well, in the third experiment, we found a solution that met all our needs, so we cancelled the rest of the experiments and moved on to other tasks…”
Minority Views on “One at a Time”

“…the factorial design has certain deficiencies … It devotes observations to exploring regions that may be of no interest…These deficiencies … suggest that an efficient design for the present purpose ought to be sequential; that is, ought to adjust the experimental program at each stage in light of the results of prior stages.”


“Some scientists do their experimental work in single steps. They hope to learn something from each run … they see and react to data more rapidly …If he has in fact found out a good deal by his methods, it must be true that the effects are at least three or four times his average random error per trial.”

Adaptive OFAT Experimentation

Empirical Evaluation of Adaptive OFAT Experimentation

- Meta-analysis of 66 responses from published, full factorial data sets
- When experimental error is <25% of the combined factor effects OR interactions are >25% of the combined factor effects, adaptive OFAT provides more improvement on average than fractional factorial DOE.

### Detailed Results

\[
\sigma = 0.1 \sqrt{MS_{FE}}
\]

\[
\sigma = 0.4 \sqrt{MS_{FE}}
\]

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<td>63/31</td>
<td>59/35</td>
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Gray if OFAT > FF
A Mathematical Model of Adaptive OFAT

initial observation

$O_0 = y(\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_n)$

observation with
first factor toggled

$O_1 = y(-\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_n)$

first factor set

$x^*_1 = \tilde{x}_1 \text{sign}\{O_0 - O_1\}$

for $i = 2 \ldots n$

repeat for all
remaining factors

$O_i = y(x^*_1, \ldots x^*_i, -\tilde{x}_i, \tilde{x}_{i+1}, \ldots \tilde{x}_n)$

$x^*_i = \tilde{x}_i \text{sign}\{\max(O_0, O_1, \ldots O_{i-1}) - O_i\}$

process ends after $n+1$ observations with

$E[y(x^*_1, x^*_2, \ldots x^*_n)]$

A Mathematical Model of a Population of Engineering Systems

\[ y(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \beta_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \beta_{ij} x_i x_j + \epsilon_k \]

- System response

- \( \beta_i \sim N(0, \sigma_{ME}^2) \) main effects
- \( \beta_{ij} \sim N(0, \sigma_{INT}^2) \) two-factor interactions
- \( \epsilon_k \sim N(0, \sigma^2) \) experimental error

- \( y_{\text{max}} \equiv \) the largest response within the space of discrete, coded, two-level factors \( x_i \in \{-1, +1\} \)

Probability of Exploiting an Effect

• The $i^{th}$ main effect is said to be “exploited” if
  \[ \beta_i x_i^* > 0 \]

• The two-factor interaction between the $i^{th}$ and $j^{th}$ factors is said to be “exploited” if
  \[ \beta_{ij} x_i^* x_j^* > 0 \]

• The probabilities and conditional probabilities of exploiting effects provide insight into the mechanisms by which a method provides improvements
The Expected Value of the Response after the *First* Step

\[ E(y(x_1^*, \tilde{x}_2, \ldots, \tilde{x}_n)) = E[\beta_1 x_1^*] + (n - 1)E[\beta_{1j} x_1^* \tilde{x}_j] \]

\[
E[\beta_1 x_1^*] = \sqrt{\frac{2}{\pi}} \frac{\sigma_{ME}^2}{\sqrt{\sigma_{ME}^2 + (n - 1)\sigma_{INT}^2 + \frac{1}{2}\sigma_{\epsilon}^2}}
\]

\[
E[\beta_{1j} x_1^* \tilde{x}_j] = \sqrt{\frac{2}{\pi}} \frac{\sigma_{INT}^2}{\sqrt{\sigma_{ME}^2 + (n - 1)\sigma_{INT}^2 + \frac{1}{2}\sigma_{\epsilon}^2}}
\]

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<th>( \sigma_{\epsilon} / \sigma_{ME} = 0.1 )</th>
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<td>Simulation</td>
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<td>( \sigma_{\epsilon} / \sigma_{ME} = 1 )</td>
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<td>Theorem 1</td>
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<td>( \sigma_{\epsilon} / \sigma_{ME} = 10 )</td>
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<td>Simulation</td>
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\( n = 7 \)
Probability of Exploiting the First Main Effect

\[
\Pr(\beta_1 x_1^* > 0) = \frac{1}{2} + \frac{1}{\pi} \sin^{-1} \left( \frac{\sigma_{ME}}{\sqrt{\sigma_{ME}^2 + (n-1)\sigma_{INT}^2 + \frac{1}{2} \sigma_{\varepsilon}^2}} \right)
\]

If interactions are small and error is not too large, OFAT will tend to exploit main effects
The Expected Value of the Response After the Second Step

\[ E(y(x_1^*, x_2^*, \tilde{x}_3, \ldots, \tilde{x}_n)) = 2E[\beta_1 x_1^*] + 2(n - 2)E[\beta_1 x_1^*] + E[\beta_{12} x_1^* x_2^*] \]

\[ E[\beta_{12} x_1^* x_2^*] = \sqrt{\frac{2}{\pi}} \left( \frac{\sigma_{\text{INT}}^2}{\sqrt{\sigma_{\text{ME}}^2 + (n - 1)\sigma_{\text{INT}}^2 + \frac{\sigma_e^2}{2}}} \right) \]
Probability of Exploiting the First Interaction

\[
\Pr(\beta_{12}x_i^*x_2^* > 0) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{\sigma_{\text{INT}}}{\sqrt{\sigma_{\text{ME}}^2 + (n-2)\sigma_{\text{INT}}^2 + \frac{1}{2}\sigma^2}}
\]

\[
\Pr(\beta_{12}x_i^*x_2^* > 0|\beta_{12} > \beta_{ij}) > \frac{1}{\pi} \int_0^{\infty} \int_{-x_2}^{x_2} \frac{\left[\text{erf} \left( \frac{1}{\sqrt{2} \sigma_{\text{INT}}} \right) \right]}{\sigma_{\text{INT}} \sqrt{\sigma_{\text{ME}}^2 + (n-2)\sigma_{\text{INT}}^2 + \frac{1}{2}\sigma^2}} \, dx_2 \, dx_1
\]
And it Continues

\[
\Pr(\beta_{ij} x_i^* x_j^* > 0) \geq \Pr(\beta_{12} x_1^* x_2^* > 0)
\]

We can prove that the probability of exploiting interactions is sustained. Further we can now prove exploitation probability is a function of \( j \) only and increases monotonically.
Final Outcome

Adaptive OFAT

Resolution III Design
Final Outcome

Adaptive OFAT

Resolution III Design
Adaptive “One Factor at a Time” for Robust Design

Run a resolution III on noise factors.

Again, run a resolution III on noise factors. If there is an improvement, in transmitted variance, retain the change.

Change one factor.

If the response gets worse, go back to the previous state.

Stop after you’ve changed every factor once.

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Image removed due to copyright restrictions. Please see Fig. 8 and 9 in Frey, D. D., and N. Sudarsanam. “An Adaptive One-factor-at-a-time Method for Robust Parameter Design: Comparison with Crossed Arrays via Case Studies.” *ASME Journal of Mechanical Design* 130 (February 2008): 021401
Results Across Four Case studies

Ensembles of aOFATs

Comparing an Ensemble of 4 aOFATs with a $2^{7-2}$ Fractional Factorial array using the HPM

Expected Value of Largest Control Factor = 16

Comparing an Ensemble of 8 aOFATs with a $2^{7-1}$ Fractional Factorial array using the HPM

Expected Value of Largest Control Factor = 16

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Conclusions

• A new model and theorems show that
  – Adaptive OFAT plans exploit two-factor interactions especially when they are large
  – Adaptive OFAT plans provide around 80% of the benefits achievable via parameter design
• Adaptive OFAT can be “crossed” with factorial designs which proves to be highly effective

Questions?