Control of
Manufacturing Processes

Subject 2.830/6.780/ESD.63
Spring 2008
Lecture #8

Process Capability &
Alternative SPC Methods

March 4, 2008
Agenda

• Control Chart Review
  – hypothesis tests: $\alpha$, $\beta$ and $n$
  – control charts: $\alpha$, $\beta$, $n$, and average run length (ARL)

• Process Capability

• Advanced Control Chart Concepts
Average Run Length

- How often will the data exceed the ±3σ limits if Δμₓ = 0?

\[
\text{Prob}(x > \mu_x + 3\sigma_x) + \text{Prob}(x < \mu_x - 3\sigma_x) = 3 / 1000
\]
Detecting Mean Shifts: Chart Sensitivity

• Consider a real shift of $\Delta \mu_x$:

• How many samples before we can expect to detect the shift on the xbar chart?
Average Run Length

• How often will the data exceed the ±3σ limits if Δμₓ = +1σ?

\[
\text{Prob}(x > \mu_x + 2\sigma_x) + \text{Prob}(x < \mu_x - 4\sigma_x) = 0.023 + 0.001 = 24 / 1000
\]
Definition

• Average Run Length (arl): Number of runs (or samples) before we can expect a limit to be exceeded = $1/p_e$

- for $\Delta \mu = 0$  $\text{arl} = 3/1000 = 333$ samples
- for $\Delta \mu = 1\sigma$  $\text{arl} = 24/1000 = 42$ samples

Even with a mean shift as large as $1\sigma$, it could take 42 samples before we know it!!!
Effect of Sample Size $n$ on ARL

- Assume the same $\Delta \mu = 1\sigma$
  - Note that $\Delta \mu$ is an absolute value

- If we increase $n$, the Variance of $x\bar{b}$ decreases:
  \[ \sigma_{x\bar{b}} = \frac{\sigma_x}{\sqrt{n}} \]

- So our $\pm 3\sigma$ limits move closer together
ARL Example

As $n$ increases $p_e$ increases so ARL decreases
Another Use of the Statistical Process Model: The Manufacturing -Design Interface

• We now have an empirical model of the process

How “good” is the process?
Is it capable of producing what we need?
Process Capability

• Assume Process is In-control
• Described fully by $xbar$ and $s$
• Compare to Design Specifications
  – Tolerances
  – Quality Loss
Design Specifications

- **Tolerances**: Upper and Lower Limits

![Diagram of Design Specifications]

- **Lower Specification Limit** ($LSL$)
- **Target** ($x^*$)
- **Upper Specification Limit** ($USL$)
Design Specifications

- **Quality Loss**: Penalty for Any Deviation from Target

\[ QLF = L^*(x-x^*)^2 \]

**How to Calibrate?**

\[ x^* = \text{target} \]
Use of Tolerances: Process Capability

- Define Process using a Normal Distribution
- Superimpose $x^*$, LSL and USL
- Evaluate Expected Performance
Process Capability

- Definitions

\[ C_p = \frac{(USL - LSL)}{6\sigma} = \frac{\text{tolerance range}}{99.97\% \text{ confidence range}} \]

- Compares ranges only
- No effect of a mean shift
Process Capability: $C_{pk}$

$$C_{pk} = \min \left( \frac{(USL - \mu)}{3\sigma}, \frac{(LSL - \mu)}{3\sigma} \right)$$

= Minimum of the normalized deviation from the mean

• Compares effect of offsets
Cp = 1; Cpk = 1
Cp = 1; Cpk = 0
Cp = 2; Cpk = 1
Cp = 2; Cpk = 2
Effect of Changes

- In Design Specs
- In Process Mean
- In Process Variance

- What are good values of Cp and Cpk?
# Cpk Table

<table>
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<th>z</th>
<th>$P&lt;\text{LS}$ or $P&gt;\text{USL}$</th>
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<td>1E-03</td>
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<tr>
<td>2</td>
<td>6</td>
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</table>
The “6 Sigma” problem

\[ P(x > 6\sigma) = 18.8 \times 10^{-10} \]

\[ C_p = 2 \]

\[ C_{pk} = 2 \]
The 6 $\sigma$ problem: Mean Shifts

$P(x>4\sigma) = 31.6 \times 10^{-6}$

Even with a mean shift of $2\sigma$ we have only 32 ppm out of spec

$C_p = 2$

$C_{pk} = 4/3$
Capability from the Quality Loss Function

QLF = L(x) = k*(x-x*)^2

Given L(x) and p(x) what is E{L(x)}?
Expected Quality Loss

\[ E\{L(x)\} = E[k(x - x^*)^2] \]

\[ = k \left[ E(x^2) - 2E(xx^*) + E(x^*^2) \right] \]

\[ = k\sigma_x^2 + k(\mu_x - x^*)^2 \]

Penalizes Variation

Penalizes Deviation
Process Capability

- The reality (the process statistics)
- The requirements (the design specs)
- Cp - a measure of variance vs. tolerance
- Cpk - a measure of variance from target
- Expected Loss - an overall measure of goodness
Xbar Chart Recap

• xbar - S (or R) charts
  – plot of sequential sample statistics
  – compare to assumptions
    • normal
    • stationary

• Interpretation
  – hypothesis tests on $\mu$ and $\sigma$
  – confidence intervals
  – “randomness”

• Application
  – Real-time decision making
Beyond Xbar

• Good Points
  – Simple and “transparent”
  – Enforces Assumptions
    • Normality (via Central Limit)
    • Independent (via long sampling times)

• Limitations
  – n>1 to get Xbar and S
  – ARL is typically large
    • Not very sensitive to small changes
  – Slow time response
Beyond Xbar

• What if n=1?
  – Have a Lot of Data
  – Want Fast Response to Changes

• How to Compute Control Chart Statistics?
  – Running Chart and Running Variance?
  – Running Average and Running Variance?
  – Running Average with Forgetting Factor

• How to Increase Sensitivity to Small, Persistent Mean Shift?
  – Integrate the Error
Chart Design:  
n=1 Designs - Running Averages

• Sensitivity: Ability to detect small changes (e.g. mean shifts)
• Time Response: Ability to Catch Changes Quickly
• Noise Rejection?: Higher Variance
Xbar “Filtering”
Filtering

- Reduced Peaks
- Hides intermediate data
- Reduces the “frequency content” of the output
Independence and Correlation

- Independence: Current output does not depend on prior
- Correlation: Measure of Independence
  - e.g. auto correlation function

\[ R_{xx}(\tau) = E[x(t)x(t + \tau)] \]
Correlation

\[ R_{xx}(\tau) = E[x(t)x(t + \tau)] \]

For a linear 1st order system

\( \tau \approx 1 \text{ sec} \)

For an uncorrelated process
Sampling: Frequency and Distribution of Samples
Correlation and Sampling

- Taking samples beyond correlation time guarantees independence
Sampling and Averaging

• Sampling Frequency Affects
  – Time Response
  – Correlation

• Averaging
  – Filters Data
  – Slows Response
Alternative Charts: Running Averages

- More averages/Data
- Can use run data alone and average for S only
- Can use to improve resolution of mean shift

\[
\bar{x}_{Rj} = \frac{1}{n} \sum_{i=j}^{j+n} x_i \quad \text{Running Average}
\]
\[
S_{Rj}^2 = \frac{1}{n-1} \sum_{i=j}^{j+n} (x_i - \bar{x}_{Rj})^2 \quad \text{Running Variance}
\]
Specific Case: Weighted Averages

\[ y_j = a_1 x_{j-1} + a_2 x_{j-2} + a_3 x_{j-3} + \ldots \]

• How should we weight measurements??
  – All equally? (as with Running Average)
  – Based on how recent?
    • e.g. Most recent are more relevant than less recent?
Consider an Exponential Weighted Average

Define a weighting function

\[ W_{t-i} = r (1 - r)^i \]
Exponentially Weighted Moving Average: (EWMA)

\[ A_i = r x_i + (1 - r) A_{i-1} \]

Recursive EWMA

\[ \sigma_A = \sqrt{\left( \frac{\sigma_x^2}{n} \right) \left( \frac{r}{2 - r} \right) \left[ 1 - (1 - r)^{2t} \right]} \]

\[ UCL, LCL = \bar{x} \pm 3 \sigma_A \]

for large \( t \)
Effect of $r$ on $\sigma$ multiplier

plot of $\frac{r}{(2-r)}$ vs. $r$

wider control limits
SO WHAT?

• The variance will be less than with xbar,

\[ \sigma_A = \frac{\sigma_x}{\sqrt{n}} \sqrt{\left( \frac{r}{2 - r} \right)} = \sigma_{\overline{x}} \sqrt{\left( \frac{r}{2 - r} \right)} \]

• n=1 case is valid

• If r=1 we have “unfiltered” data
  – Run data stays run data
  – Sequential averages remain

• If r<<1 we get long weighting and long delays
  – “Stronger” filter; longer response time
EWMA vs. Xbar

\[ r = 0.3 \]
\[ \Delta \mu = 0.5 \sigma \]
Mean Shift Sensitivity
EWMA and Xbar comparison

Mean shift = .5 σ

\( n=5 \)
\( r=0.1 \)
Effect of $r$

$r=0.3$
Small Mean Shifts

• What if $\Delta \mu_x$ is small wrt $\sigma_x$?

• But it is “persistent”

• How could we detect?
  – ARL for xbar would be too large
Another Approach: Cumulative Sums

• Add up deviations from mean
  – A Discrete Time Integrator

\[ C_j = \sum_{i=1}^{j} (x_i - \bar{x}) \]

• Since \( E\{x-\mu\} = 0 \) this sum should stay near zero
• Any bias in \( x \) will show as a trend
Mean Shift Sensitivity: CUSUM

\[ C_i = \sum_{i=1}^{t} (x_i - \bar{x}) \]

Mean shift = 1\(\sigma\)

Slope cause by mean shift \(\Delta \mu\)
Control Limits for CUSUM

- Significance of Slope Changes?
  - Detecting Mean Shifts
- Use of v-mask
  - Slope Test with Deadband

\[ d = \frac{2}{\delta} \ln \left( \frac{1 - \beta}{\alpha} \right) \]
\[ \delta = \frac{\Delta \bar{x}}{\sigma_{\bar{x}}} \]
\[ \theta = \tan^{-1} \left( \frac{\Delta \bar{x}}{2k} \right) \]

where
- \( k = \text{horizontal scale factor for plot} \)
Use of Mask

\[ \theta = \tan^{-1}(\Delta\mu/2k) \]

k = 4:1; \( \Delta\mu = 0.25 \) \((1\sigma)\)

\[ \tan(\theta) = 0.5 \text{ as plotted} \]
An Alternative

• Define the Normalized Statistic
  \[ Z_i = \frac{X_i - \mu_x}{\sigma_x} \]
  Which has an expected mean of 0 and variance of 1

• And the CUSUM statistic
  \[ S_i = \sum_{i=1}^{t} Z_i \]
  \[ S_i = \frac{\sum_{i=1}^{t} Z_i}{\sqrt{t}} \]
  Which has an expected mean of 0 and variance of 1

Chart with Centerline = 0 and Limits = ±3
Example for Mean Shift = $1\sigma$
Tabular CUSUM

- Create Threshold Variables:

\[
C_i^+ = \max[0, x_i - (\mu_0 + K) + C_{i-1}^+] \\
C_i^- = \max[0, (\mu_0 - K) - x_i + C_{i-1}^-]
\]

Accumulates deviations from the mean

\[K = \frac{|\Delta \mu|}{2} \]

\[\Delta \mu = \text{mean shift to detect}\]

\[H : \text{alarm level (typically 5} \sigma\)\]
Threshold Plot

- $\mu = 0.495$
- $\sigma = 0.170$
- $k=\frac{\delta \mu}{2} = 0.049$
- $h=5\sigma = 0.848$
Alternative Charts Summary

• Noisy Data Need Some Filtering
• Sampling Strategy Can Guarantee Independence
• Linear Discrete Filters have Been Proposed
  – EWMA
  – Running Integrator
• Choice Depends on Nature of Process
Summary of SPC

• Consider Process a Random Process
  – Can never predict precise value

• Model with $P(x)$ or $p(x)$
  – Assume $p(x,t) = p(x)$

• Shewhart Hypothesis
  – In-control = purely random output
    • Normal, independent stationary
    • “The best you can do!”
  – Not in-control
    • Non-random behavior
    • Source can be found and eliminated
The SPC Hypothesis

In-Control

Not In-Control

$p(y)$

Process