MIT 2.853/2.854
Introduction to Manufacturing Systems

Probability

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I flip a coin 100 times, and it shows heads every time.

Question: What is the probability that it will show heads on the next flip?
I flip a coin 100 times, and it shows heads every time.

Question:  *How much would you bet* that it will show heads on the next flip?
I flip a coin 100 times, and it shows heads every time.

**Question:** What odds would you demand before you **bet** that it will show heads on the next flip?
Probability and Statistics

Probability $\neq$ Statistics

**Probability:** mathematical theory that describes uncertainty.

**Statistics:** set of techniques for extracting useful information from data.
Interpretations of probability

Frequency

The probability that the outcome of an experiment is $A$ is $P(A)$...

if the experiment is performed a large number of times and the fraction of times that the observed outcome is $A$ is $P(A)$. 
Interpretations of probability

Parallel universes

The probability that the outcome of an experiment is A is $P(A)$...

if the experiment is performed in each parallel universe and the fraction of universes in which the observed outcome is A is $P(A)$. 
Interpretations of probability

Betting odds

The probability that the outcome of an experiment is A is $P(A)$...

if before the experiment is performed a risk-neutral observer would be willing to bet $1$ against more than $\frac{1-P(A)}{P(A)}$.

The expected value (slide 35) of the bet is greater than

$$(1 - P(A)) \times (-1) + P(A) \times \frac{1 - P(A)}{P(A)} = 0$$
Interpretations of probability

State of belief

*The probability that the outcome of an experiment is A is* \( P(A) \)...

if that is the opinion (ie, belief or state of mind) of an observer *before* the experiment is performed.
Interpretations of probability

Abstract measure

*The probability that the outcome of an experiment is A is* $P(A)$...

if $P()$ satisfies a certain set of conditions: *the axioms of probability.*
Interpretations of probability

Axioms of probability

Let \( U \) be a set of \textit{samples}. Let \( E_1, E_2, \ldots \) be subsets of \( U \).

Let \( \emptyset \) be the \textit{null} (or \textit{empty}) \textit{set}, the set that has no elements.

- \( 0 \leq P(E_i) \leq 1 \)
- \( P(U) = 1 \)
- \( P(\emptyset) = 0 \)
- If \( E_i \cap E_j = \emptyset \), then \( P(E_i \cup E_j) = P(E_i) + P(E_j) \)
Probability Basics
Discrete Sample Space

Notation, terminology:

- $\omega$ is often used as the symbol for a generic sample.
- Subsets of $U$ are called *events*.
- $P(E)$ is the *probability* of $E$. 
Probability Basics
Discrete Sample Space

- **Example:** Throw a single die. The possible outcomes are \{1, 2, 3, 4, 5, 6\}. \(\omega\) can be any one of those values.

- **Example:** Consider \(n(t)\), the number of parts in inventory at time \(t\). Then

\[
\omega = \{n(1), n(2), \ldots, n(t), \ldots\}
\]

is a sample path.
Probability Basics
Discrete Sample Space

- An event can often be defined by a statement. For example,

\[ \mathcal{E} = \{ \text{There are 6 parts in the buffer at time } t = 12 \} \]

Formally, this can be written

\[ \mathcal{E} = \{ \omega \mid n(12) = 6 \} \]
Probability Basics
Discrete Sample Space

High probability

Low probability
Probability Basics

Set Theory

Venn diagrams

\[ P(\bar{A}) = 1 - P(A) \]
Venn diagrams

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
A and $B$ are \textit{independent} if

$$P(A \cap B) = P(A)P(B).$$
If $P(B) \neq 0$, 

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We can also write $P(A \cap B) = P(A|B)P(B)$. 
Conditional Probability

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \]

**Example:** Throw a die. Let

- \( A \) is the event of getting an odd number (1, 3, 5).
- \( B \) is the event of getting a number less than or equal to 3 (1, 2, 3).

Then \( P(A) = P(B) = 1/2, P(A \cap B) = P(1, 3) = 1/3. \)

Also, \( P(A \mid B) = P(A \cap B)/P(B) = 2/3. \)
Let $B = C \cup D$ and assume $C \cap D = \emptyset$. Then

$$P(A|C) = \frac{P(A \cap C)}{P(C)} \quad \text{and} \quad P(A|D) = \frac{P(A \cap D)}{P(D)}.$$
Probability Basics

Law of Total Probability

Also,

- \[ P(C|B) = \frac{P(C \cap B)}{P(B)} = \frac{P(C)}{P(B)} \]  because \( C \cap B = C \).

Similarly, \[ P(D|B) = \frac{P(D)}{P(B)} \]

- \[ A \cap B = A \cap (C \cup D) = (A \cap C) \cup P(A \cap D) \]

- Therefore

  \[ P(A \cap B) = P(A \cap (C \cup D)) \]

  \[ = P(A \cap C) + P(A \cap D) \]  because \( (A \cap C) \) and \( (A \cap D) \) are disjoint.
Law of Total Probability

- Or, \( P(A|B)P(B) = P(A|C)P(C) + P(A|D)P(D) \)

or,

\[
\frac{P(A|B)P(B)}{P(B)} = \frac{P(A|C)P(C)}{P(B)} + \frac{P(A|D)P(D)}{P(B)}
\]

or,

\[
P(A|B) = P(A|C)P(C|B) + P(A|D)P(D|B)
\]
An important case is when $C \cup D = B = U$, so that $A \cap B = A$. Then $P(A) = P(A \cap C) + P(A \cap D)$ or

$$P(A) = P(A|C)P(C) + P(A|D)P(D)$$
More generally, if $A$ and $\mathcal{E}_1, \ldots, \mathcal{E}_k$ are events and

$$\mathcal{E}_i \text{ and } \mathcal{E}_j = \emptyset, \text{ for all } i \neq j$$

and

$$\bigcup_{j} \mathcal{E}_j = \text{ the universal set}$$

(i.e., the set of $\mathcal{E}_j$ sets is \textit{mutually exclusive} and \textit{collectively exhaustive}) then ...
Probability Basics
Law of Total Probability

\[ \sum_j P(\mathcal{E}_j) = 1 \]

and

\[ P(A) = \sum_j P(A|\mathcal{E}_j)P(\mathcal{E}_j). \]
Example

\( A = \{ \text{I will have a cold tomorrow.} \} \)
\( \mathcal{E}_1 = \{ \text{It is raining today.} \} \)
\( \mathcal{E}_2 = \{ \text{It is snowing today.} \} \)
\( \mathcal{E}_3 = \{ \text{It is sunny today.} \} \)

(Assume \( \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 = U \) and \( \mathcal{E}_1 \cap \mathcal{E}_2 = \mathcal{E}_1 \cap \mathcal{E}_3 = \mathcal{E}_2 \cap \mathcal{E}_3 = \emptyset \).)

Then \( A \cap \mathcal{E}_1 = \{ \text{I will have a cold tomorrow and it is raining today} \} \).
And \( P(A|\mathcal{E}_1) \) is the probability I will have a cold tomorrow \textit{given} that it is raining today.

etc.
Then

\{I \text{ will have a cold tomorrow.}\} =
\{I \text{ will have a cold tomorrow and it is raining today}\} \cup
\{I \text{ will have a cold tomorrow and it is snowing today}\} \cup
\{I \text{ will have a cold tomorrow and it is sunny today}\}

so

\[ P(\{I \text{ will have a cold tomorrow.}\}) =
P(\{I \text{ will have a cold tomorrow and it is raining today}\}) +
P(\{I \text{ will have a cold tomorrow and it is snowing today}\}) +
P(\{I \text{ will have a cold tomorrow and it is sunny today}\}) \]
Probability Basics
Law of Total Probability

\[ P(\{\text{I will have a cold tomorrow.}\}) = \]
\[ P(\{\text{I will have a cold tomorrow} \mid \text{it is raining today}\}) P(\{\text{it is raining today}\}) + \]
\[ P(\{\text{I will have a cold tomorrow} \mid \text{it is snowing today}\}) P(\{\text{it is snowing today}\}) + \]
\[ P(\{\text{I will have a cold tomorrow} \mid \text{it is sunny today}\}) P(\{\text{it is sunny today}\}) \]

or

\[ P(A) = P(A \mid \mathcal{E}_1) P(\mathcal{E}_1) + P(A \mid \mathcal{E}_2) P(\mathcal{E}_2) + P(A \mid \mathcal{E}_3) P(\mathcal{E}_3) \]
Let $V$ be a vector space. Then a random variable $X$ is a mapping (a function) from $U$ to $V$.

If $\omega \in U$ and $x = X(\omega) \in V$, then $X$ is a random variable.

**Example:** $V$ could be the real number line.

**Typical notation:**

- Upper case letters ($X$) are usually used for random variables and corresponding lower case letters ($x$) are usually used for possible values of random variables.
- Random variables ($X(\omega)$) are usually not written as functions; the argument ($\omega$) of the random variable is usually not written. *This sometimes causes confusion.*
Flip of a Coin

Let $U=\text{H,T}$. Let $\omega = \text{H}$ if we flip a coin and get heads; $\omega = \text{T}$ if we flip a coin and get tails.

Let $V$ be the real number line. Let $X(\omega)$ be the number of times we get heads. Then $X(\omega) = 0$ or $1$.

Assume the coin is fair. (*No tricks this time!*)

Then

$P(\omega = \text{T} ) = P(X = 0) = 1/2$

$P(\omega = \text{H} ) = P(X = 1) = 1/2$
Flip of Three Coins

Let $U=$ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.

Let $\omega =$ HHH if we flip 3 coins and get 3 heads; $\omega =$ HHT if we flip 3 coins and get 2 heads and then one tail, etc. The order matters! There are 8 samples.

- $P(\omega) = 1/8$ for all $\omega$.

Let $X$ be the number of heads. Then $X = 0, 1, 2, \text{ or } 3$.

- $P(X = 0) = 1/8; P(X = 1) = 3/8; P(X = 2) = 3/8; P(X = 3) = 1/8.$

There are 4 distinct values of $X$. 
Let $X(\omega)$ be a random variable. Then $P(X(\omega) = x)$ is the \textit{probability distribution} of $X$ (usually written $P(x)$). For three coin flips:

\begin{center}
\begin{tabular}{c|c|c|c|c}
\hline
$x$ & 0 & 1 & 2 & 3 \\
\hline
$P(x)$ & $1/8$ & $3/8$ & $1/4$ & $1/8$ \\
\hline
\end{tabular}
\end{center}
Mean and Variance

Mean (average): \( \bar{x} = \mu_x = E(X) = \sum_x x P(x) \)

Variance: \( V_x = \sigma_x^2 = E(x - \mu_x)^2 = \sum_x (x - \mu_x)^2 P(x) \)

Standard deviation: \( \sigma_x = \sqrt{V_x} \)

Coefficient of variation (cv): \( \sigma_x / \mu_x \)
For three coin flips:

\[ \bar{x} = 1.5 \]
\[ V_x = 0.75 \]
\[ \sigma_x = 0.866 \]
\[ cv = 0.577 \]
Probability Basics
Functions of a Random Variable

- A function of a random variable is a random variable.

- Special case: linear function

  For every \( \omega \), let \( Y(\omega) = aX(\omega) + b \). Then

  \[
  \begin{align*}
  \bar{Y} &= a \bar{X} + b, \\
  V_Y &= a^2 V_X; \\
  \sigma_Y &= |a| \sigma_X.
  \end{align*}
  \]
Covariance

$X$ and $Y$ are random variables. Define the covariance of $X$ and $Y$ as:

$$\text{Cov}(X, Y) = E \left[ (X - \mu_x)(Y - \mu_y) \right]$$

Facts:

- $\text{Var}(X + Y) = V_x + V_y + 2\text{Cov}(X, Y)$
- If $X$ and $Y$ are independent, $\text{Cov}(X, Y) = 0$.
- If $X$ and $Y$ vary in the same direction, $\text{Cov}(X, Y) > 0$.
- If $X$ and $Y$ vary in the opposite direction, $\text{Cov}(X, Y) < 0$. 
The *correlation* of $X$ and $Y$ is

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_x \sigma_y}$$

$$-1 \leq \text{Corr}(X, Y) \leq 1$$
Flip a biased coin. Assume all flips are independent.

$X^B$ is 1 if outcome is heads; 0 if tails.

$P(X^B = 1) = p$.

$P(X^B = 0) = 1 - p$.

$X^B$ is Bernoulli.
Discrete Random Variables

Binomial

The sum of \( n \) independent Bernoulli random variables \( X_i^B \) with the same parameter \( p \) is a \textit{binomial} random variable \( X^b \).

\[
X^b = \sum_{i=0}^{n} X_i^B
\]

\[
P(X^b = x) = \frac{n!}{x!(n-x)!} p^x (1 - p)^{(n-x)}
\]
Discrete Random Variables

Binomial probability distribution

$\text{N=100, p=0.4}$
The number of independent Bernoulli random variables $X_i^B$ with the same parameter $p$ tested until the first 1 appears is a geometrically distributed random variable $X^g$.

$$X^g = k \text{ if } X_1^B = 0, X_2^B = 0, \ldots, X_{k-1}^B = 0, X_k^B = 1$$
Discrete Random Variables

Geometric

To calculate $P(X^g = k)$, recall that $P(X^g = 1) = p$, so $P(X^g > 1) = 1 - p$.

Then

$$P(X^g > k) = P(X^g > k | X^g > k - 1)P(X^g > k - 1)$$

$$= (1 - p)P(X^g > k - 1),$$

because

$$P(X^g > k | X^g > k - 1) = P(X^B_1 = 0, ..., X^B_k = 0 | X^B_1 = 0, ..., X^B_{k-1} = 0) = 1 - p$$

so

$$P(X^g > 1) = 1 - p, \ P(X^g > 2) = (1 - p)^2, \ ... \ P(X^g > k - 1) = (1 - p)^{k-1}$$

and $P(X^g = k) = P(\{X^g > k - 1\} \text{ and } \{X^B_k = 1\}) = (1 - p)^{k-1}p$. 

Probability

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Discrete Random Variables

Geometric probability distribution

Geometric, \( p = 0.1 \)
Discrete Random Variables

Poisson Distribution

\[ P(X^P = x) = e^{-\lambda} \frac{\lambda^x}{x!} \]

Discussion later.
1. *Mathematically*, continuous and discrete random variables are very different.

2. *Quantitatively*, however, some continuous models are very close to some discrete models.

3. Therefore, which kind of model to use for a given system is a matter of *convenience*.
Example: The production process for small metal parts (nuts, bolts, washers, etc.) might better be modeled as a continuous flow than as a large number of discrete parts.
The probability of a two-dimensional random variable being in a small square is the *probability density* times the area of the square. (The definition is similar in higher-dimensional spaces.)
Continuous Random Variables
Philosophical Issues
Continuous Random Variables

Spaces

Dimensionality

- Continuous random variables can be defined
  - in one, two, three, ..., infinite dimensional spaces;
  - in finite or infinite regions of the spaces.

- Continuous random variables can have
  - probability measures with the same dimensionality as the space;
  - lower dimensionality than the space;
  - a mix of dimensions.
Continuous Random Variables
No change in water levels
Continuous Random Variables

One kind of change in water levels
Continuous Random Variables

Two-dimensional probability distribution

One-dimensional density

Two-dimensional density

Zero-dimensional density (mass)

Probability distribution of the amount of material in each of the two buffers.
Continuous Random Variables

Trajectories

Trajectories of buffer levels in the three-machine line if the machine states stay constant for a long enough time period.
Continuous Random Variables
Discrete approximation of the probability distribution

Probability distribution of the amount of material in each of the two buffers.
In one dimension, $F()$ is the *cumulative probability distribution of $X$* if

$$F(x) = P(X \leq x)$$

$f()$ is the *density function of $X$* if

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

or

$$f(x) = \frac{dF}{dx}$$

wherever $F$ is differentiable.
Continuous Random Variables
Densities and Distributions

Fact: \( F(b) - F(a) = \int_a^b f(t) \, dt \)

Fact: \( f(x) \delta x \approx P(x \leq X \leq x + \delta x) \) for sufficiently small \( \delta x \).

Definition: \( \bar{x} = \int_{-\infty}^{\infty} tf(t) \, dt \)
Continuous Random Variables

Law of Total Probability

Scalar version

\[ f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)f_Y(y)\,dy \]

This is also extended to more dimensions.
Continuous Random Variables

Normal Distribution

The density function of the normal (or gaussian) distribution with mean 0 and variance 1 (the standard normal) is given by

\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \]

The normal distribution function is

\[ F(x) = \int_{-\infty}^{x} f(t)dt \]

(There is no closed form expression for \( F(x) \).)
Continuous Random Variables

Normal Distribution
Continuous Random Variables

Normal Distribution

Notation: \( N(\mu, \sigma) \) is the normal distribution with mean \( \mu \) and variance \( \sigma^2 \).

Note: Some people write \( N(\mu, \sigma^2) \) for the normal distribution with mean \( \mu \) and variance \( \sigma^2 \).

Fact: If \( X \) and \( Y \) are normal, then \( aX + bY + c \) is normal.

Fact: If \( X \) is \( N(\mu, \sigma) \), then \( \frac{X - \mu}{\sigma} \) is \( N(0, 1) \), the standard normal.

This is why \( N(0, 1) \) is tabulated in books and why \( N(\mu, \sigma) \) is easy to compute from \( N(0, 1) \).
Continuous Random Variables
Truncated Normal Density

\[ P(x \leq X \leq x + \delta x) = \frac{f(x)}{1 - F(0)} \delta x \]

where \( F() \) and \( f() \) are the normal distribution and density functions with parameters \( \mu \) and \( \sigma \).
Continuous Random Variables
Another Kind of Truncated Normal Density

\[ P(x \leq X \leq x + \delta x) = f(x) \delta x \text{ for } x > 0 \text{ and } P(X = 0) = F(0) \text{ where } F() \text{ and } f() \text{ are the normal distribution and density functions with parameters } \mu \text{ and } \sigma. \]
Let \( \{X_k\} \) be a sequence of independent identically distributed (i.i.d.) random variables that have the same finite mean \( \mu \). Let \( S_n \) be the sum of the first \( n \) \( X_k \)s, so

\[
S_n = X_1 + \ldots + X_n
\]

Then for every \( \epsilon > 0 \),

\[
\lim_{n \to \infty} P \left( \left| \frac{S_n}{n} - \mu \right| > \epsilon \right) = 0
\]

That is, the average approaches the mean.
Continuous Random Variables

Central Limit Theorem

Let \( \{X_k\} \) be a sequence of i.i.d. random variables with finite mean \( \mu \) and finite variance \( \sigma^2 \).

Then as \( n \to \infty \), \( P\left( \frac{S_n - n\mu}{\sqrt{n}\sigma} \right) \to N(0, 1) \).

If we define \( A_n \) as \( S_n/n \), the average of the first \( n \) \( X_k \)s, then this is equivalent to:

As \( n \to \infty \), \( P(A_n) \to N(\mu, \sigma/\sqrt{n}) \).
Continuous Random Variables
Coin flip examples

Probability of $x$ heads in $n$ flips of a fair coin

Probability (n=3)

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<td>29</td>
<td>0.00</td>
</tr>
<tr>
<td>30</td>
<td>0.00</td>
</tr>
</tbody>
</table>

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Continuous Random Variables

Binomial probability distribution approaches normal for large $N$. 

N=100, $p=0.4$
Continuous Random Variables
Binomial distributions

Note the resemblance to a *truncated* normal in these examples.
Normal Density Function

... in Two Dimensions
More Continuous Distributions

Uniform

\[ f(x) = \frac{1}{b - a} \quad \text{for } a \leq x \leq b \]

\[ f(x) = 0 \quad \text{otherwise} \]
More Continuous Distributions

Uniform

Uniform density

Uniform distribution
More Continuous Distributions

Triangular

Probability density function

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More Continuous Distributions

Triangular

Cumulative distribution function

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More Continuous Distributions

Exponential

- $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$; $f(t) = 0$ otherwise;
  $P(T > t) = e^{-\lambda t}$ for $t \geq 0$; $P(T > t) = 1$ otherwise.

- Close to the geometric distribution but for continuous time.

- *Very* mathematically convenient. Often used as model for the first time until an event occurs.

- Memorylessness:
  $P(T > t + x | T > x) = P(T > t)$

The cumulative probability distribution

$F(t) = 1 - P(T > t) = 1 - e^{-\lambda t}$ for $t \geq 0$; $F(t) = 0$ otherwise.
More Continuous Distributions

Exponential distribution

Exponential density

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Discrete Random Variables

Poisson Distribution

\[ P(X^P = x) = e^{-\lambda t} \frac{(\lambda t)^x}{x!} \]

is the probability that \( x \) events happen in \([0, t]\) if the events are independent and the times between them are exponentially distributed with parameter \( \lambda \).

Typical examples: arrivals and services at queues. (Next lecture!)
A pseudo-random number generator is a set of numbers $X_0, X_1, \ldots$ where there is a function $F$ such that

$$X_{n+1} = F(X_n)$$

and $F$ is such that the sequence of $X_n$ satisfies certain conditions.

For example $0 \leq X_n \leq 1$ and the sequence $X_0, X_1, \ldots$ looks like uniformly distributed, independent random variables.

That is, statistical tests say that the probability of the sequence not being independent uniform random variables is very small.

However the sequence is deterministic: it is determined by $X_0$, the seed of the random number generator.

Pseudo-random number generators are used extensively in simulation.