What we’ll do today

• Information content for multi-FR
  – Basic statistics/probability

• Allowable tolerance (linear tolerancing) vs. statistical tolerancing
Information content

\[ P(\text{FR}) = \int_{\text{dr}^l}^{\text{dr}^u} f(\text{FR})d\text{FR} \]

\[ I = -\log_2 P = -\log_2 P(\text{FR}) = -\log_2 \int_{\text{dr}^l}^{\text{dr}^u} f(\text{FR})d\text{FR} \]
Normal Distribution

We want to know this area (probability)

\( X \sim N(\mu, \sigma^2) \)

Then, \( Z = (X-\mu)/\sigma \sim N(0, 1) \)

<table>
<thead>
<tr>
<th>z</th>
<th>( \phi(z) )</th>
<th>1-( \phi(z) )</th>
<th>prob between +z, -z</th>
<th>ppm</th>
<th>ppb</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.84134474</td>
<td>0.15865526</td>
<td>0.68268948</td>
<td>317310.5</td>
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<tr>
<td>2</td>
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<td>0.022750062</td>
<td>0.954499876</td>
<td>45500.12</td>
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<td>3</td>
<td>0.998650033</td>
<td>0.001349967</td>
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<td>4</td>
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<td>63.37207</td>
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<tr>
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<td>0.999999998</td>
<td>0.00198</td>
<td>1.980244</td>
</tr>
</tbody>
</table>
HW1, #5

- System range, \( FR_1 \sim N(\mu,\sigma^2) \)
- Design range \( dr_i \leq FR_1 \leq dr_u \)  
  Q: Information Content?
- \( \lambda = (dr_u + dr_i)/2 - \mu \)

In terms of \( \sigma \) multiple: 
\[
\frac{dr^l - (dr^l + dr^u)/2 + \lambda}{\sigma} \quad \frac{dr^u - (dr^l + dr^u)/2 + \lambda}{\sigma}
\]
Multiple FR system range

Example

\[
\begin{align*}
\{FR1\} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \{DP1\} \\
\{FR2\} &= \begin{bmatrix} 1 & 1 \end{bmatrix} \{DP2\}
\end{align*}
\]

Design range

FR1: [-0.5, 0.5]
FR2: [-2.0, 2.0]

Assuming statistical independence between DP1 and DP2, the joint pdf of (DP1,DP2) is a product of pdf(DP1) and pdf(DP2).

DP variation (joint pdf) is mapped onto FR space; only a projection is shown here.

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Detecting change in system range

“Monitoring marginal probability of each FR is not only inaccurate but potentially misleading”

Example

\[
\begin{align*}
\{FR1\} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \{DP1\} \\
\{FR2\} &= \begin{bmatrix} 1 & 1 \end{bmatrix} \{DP2\}
\end{align*}
\]

Design range

FR1: [-0.5, 0.5]  
FR2: [-2, 2]

Design parameter variation

Initial  
DP1: U[-1,1]  
DP2: U[0,1.5]  

After change  
DP1: U[-1,1]  
DP2: U[-1,1.6]

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Before DP2 change

<table>
<thead>
<tr>
<th></th>
<th>Wrong</th>
<th>Correct</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p_{FR1}$</td>
<td>$p_{FR2}$</td>
</tr>
<tr>
<td>Before</td>
<td>0.5</td>
<td>0.9583</td>
</tr>
<tr>
<td>After</td>
<td>0.5</td>
<td>0.9654</td>
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</tbody>
</table>
Allowable tolerance

- Defined for DP
- Tolerances that DPs can take while FRs still remaining completely inside design ranges
- Unconditional tolerance
- Conservative tolerancing

\[
\Delta D_{P1} = \frac{\Delta F_{R1}}{A_{11}} \\
\Delta D_{P2} = \frac{\Delta F_{R2} - |A_{21} \cdot \Delta D_{P1}|}{A_{22}}
\]
Linear tolerancing vs. Statistical tolerancing

\[
\begin{bmatrix}
FR1 \\
FR2
\end{bmatrix} = 
\begin{bmatrix}
1 & 0 \\
0.4 & 1
\end{bmatrix}
\begin{bmatrix}
DP1 \\
DP2
\end{bmatrix}
\]

Design range

FR1: [-0.6, 0.6] 
FR2: [-1.8, 1.8]

Statistical tolerancing

\[3\sigma_{F_{R1}} = 0.6 \Rightarrow \sigma_{F_{R1}} = 0.2\]

Therefore, \(\sigma_{D_{P1}} = 0.2\)

\[\text{Var}(F_{R2}) = 0.4^2 \text{Var}(D_{P1}) + 1^2 \text{Var}(D_{P2})\]

Thus, \(\sigma_{D_{P2}} = 0.5946\)

\[3 \sigma_{D_{P1}} = 0.6\]
\[3 \sigma_{D_{P2}} = 1.784\]