1 Adiabatic Quantum Computation

Adiabatic quantum computation is a Hamiltonian-Based model of quantum computation. In quantum mechanics, the Schrödinger equation gives the time-evolution of a state $|\psi\rangle$ in terms of the Hamiltonian operator $H$,

$$H|\psi\rangle = i\hbar \frac{\partial}{\partial t} |\psi\rangle.$$ 

Furthermore, if $|\psi\rangle$ is an energy eigenstate with energy $E$,

$$|\psi(t)\rangle = e^{-iEt/\hbar} |\psi(0)\rangle.$$ 

The idea behind Hamiltonian-based QC is that finding the ground state of a Hamiltonian solves interesting computational problems, including NP complete problems.

**Adiabatic Theorem**

The adiabatic theorem says that if we begin in the ground state and change the Hamiltonian slowly, be end in the ground state of the new Hamiltonian. Consider the Hamiltonian

$$H(t) \text{ for } 0 \leq t \leq 1.$$ 

Where do we end up if we are in the ground state at $s = 0$?

$$H_T(t) = H\left(\frac{t}{T}\right)$$ 

$$H_T|\psi\rangle = i\hbar \frac{\partial}{\partial t} |\psi\rangle$$ 

$$|\psi_{t=0}\rangle = |\text{ground state}\rangle$$ 

**Theorem.** As $T \to \infty$, $|\psi_{t=\tau}\rangle \to |\text{ground state}\rangle$. 

**How Slow Must the Hamiltonian Change?**

The time depends on the energy gap $\Delta$ between the ground state and the lowest excited state.

"Theorem". If $T \propto 1/\Delta^2$, we stay in the ground state (no rigorous proof known).

**Theorem.** If $T \propto 1/\Delta^3$, we stay in the ground state.

**Theorem.** If $T \propto 1/\Delta^2$ and the Hamiltonian is smooth, we stay in the ground state.

**Example: Grover’s Algorithm**

Grover’s algorithm finds a unique goal state,

$$H|\text{goal}\rangle = 0$$ 

$$H|s, s \neq \text{goal}\rangle = 1.$$ 

We begin with the state:

$$\left[ \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \right]^\otimes n,$$
which is the ground state of the Hamiltonian,

\[ H = -\sigma_x^{(1)} - \sigma_x^{(2)} - \ldots - \sigma_x^{(n)}. \]

To find the gap, let \(|\text{goal}\rangle = |00\ldots0\rangle\). Note that the beginning state is symmetric: exchanging any two qubits yields the same Hamiltonian. The space of symmetric states has dimensionality \(n+1\) and basis states: \(\frac{1}{\binom{n}{k}} \sum |s_k\rangle\), where the sum is over all bit strings with \(k\) 0s. The smallest gap is \(\mathcal{O}(1/\sqrt{2^n})\) so the running time is \(\mathcal{O}(2^n)\). However, the gap over time looks like:

so by moving quickly when the gap is large, the time becomes \(\mathcal{O}(\sqrt{2^n})\).

**Example: 3-SAT**

3-SAT may be expressed as finding the ground state of a Hamiltonian. Given \(n\) variables with possible values 0 and 1, and \(m\) clauses of the form \(x_i \lor \overline{x_j} \lor x_k\) (with any number of negations), can all clauses be satisfied for some assignment of the variables? The Hamiltonian that answers this question is:

\[ H_C = (1 + Z_i)(1 - Z_j)(1 + Z_k) \]

\[ H = \sum_{i=1}^{m} H_C \]

Note: the Hamiltonian is 3-local, which is desireable for physical implementations. What is the gap for random instances of 3-SAT? Nobody knows.

## 2 Universality of Adiabatic QC

**Theorem.** Any quantum computation is equivalent, up to polynomial factors to 5-local adiabatic quantum computation.

Steps:

1. Find a ground state that accomplishes the computation.

2. Show that the gap is large.

**Encoding a Computation**

Given a computation on \(s\) qubits with \(t\) steps, step \(i\) is given by:

\[ U_i |\psi_{i-1}\rangle = |\psi_i\rangle \]

\[ |\psi_0\rangle = |00\ldots0\rangle \]
The first bit encodes the answer. We want the probability of the first bit being 1 to be close to 1 or close to 0. There are two registers: computation and clock. The ground state that encodes the computation is given by:

\[
W_i = U_i U_{i-1} \ldots U_1
\]

\[
|\text{g.s.}\rangle = \frac{1}{\sqrt{t+1}} \sum_{k=0}^{t} W_k |0\rangle_{\text{comp}} \otimes |k\rangle_{\text{clock}}
\]

**Energy Penalties**

We want to assign energy penalties for:
- having an incorrect initial state
- incorrect evolution
- bit 1 \(\neq |0\rangle\) in final state.

For the initial state we use a term picking out \(|1\rangle_k |0\rangle_{\text{clock}}\):

\[
E|1\rangle_k |0\rangle_{\text{clock}} \langle 1| |0\rangle_{\text{clock}}.
\]

A similar term can be used for the final state.

When the clock goes from \(k-1\) to \(k\), we need to apply \(U_k\) to the state. The penalty term to accomplish this is:

\[
I \otimes |k\rangle \langle k| - U_k \otimes |k\rangle \langle k-1| - U_k^\dagger \otimes |k-1\rangle |k\rangle + I \otimes |k-1\rangle \langle k-1|.
\]

The Hamiltonian’s ground state thus encodes the entire evolution.

**Modified 3-Local Hamiltonian**

To make the Hamiltonian 3-local, we represent the clock in unary:

\[
\text{time } k: |1\rangle^k |0\rangle^{n-k}.
\]

Not all clock states are valid in this representation. We need to ensure there are no \(|01\rangle\) sequences in the clock states. We can now use 3-qubit clock terms in the penalty term for evolution.

**Energy Gap**

- Have: \(H|\psi\rangle \approx 0\) if the computation answer is “yes”.
- Need: \(H|\psi\rangle > 1/n^k\) if the answer is “no”.
- Need: excited state \(H|\psi\rangle > 1/n^k\) if the answer is “yes”.

We perform a change of basis:

\[
W_k |0\rangle_{\text{comp}} |k\rangle_{\text{clock}} \to |0\rangle_{\text{comp}} |k\rangle_{\text{clock}}
\]

accomplished by applying the unitary:

\[
\sum_k W_k^\dagger \otimes |k\rangle \langle k|
\]

to the original ground state. The effect on the Hamiltonian is:

\[
U_k \otimes |k\rangle \langle k-1| \to I \otimes |k\rangle \langle k-1|,
\]

making the gap simple to calculate.
3 Fault Tolerance

Considering the final Hamiltonian, there may be an additional error term:

\[ H_{\text{final}} \rightarrow \text{ground state } |\psi\rangle \]
\[ H_{\text{final}} + H_{\text{error}} \rightarrow |\tilde{\psi}\rangle \]

We encode each qubit \( \{0, 1\} \) to \( \{0_L, 1_L\} \):

\[
\begin{align*}
|0\rangle & \rightarrow \frac{1}{2} (|0000\rangle + i|0011\rangle + i|1100\rangle + |1111\rangle) \\
|1\rangle & \rightarrow \frac{1}{2} (-|0101\rangle + i|0110\rangle + |1001\rangle - |1010\rangle)
\end{align*}
\]

These states are in a subspace stabilized by:

\[
\begin{align*}
XXXX \psi \\
ZZZZ \psi \\
XYZI \psi 
\end{align*}
\]

so we assign an energy penalty for not lying in the logical subspace:

\[ H_p = \sum -E_p \left( g_1^{(k)} + g_2^{(k)} + g_3^{(k)} \right). \]

One-local errors take logical states outside the subspace. We want to put the Hamiltonian in terms of \( X_L, Y_L, Z_L \):

\[
\begin{align*}
X_L & = YIYI \\
Y_L & = -IXXI \\
Z_L & = ZZII
\end{align*}
\]

Note that this takes a \( k \)-local Hamiltonian to a \( 2k \)-local one. A 5 qubit code protects against 2-local noise implemented with 3-local operators.

**Condition.** \( G = \langle g_1, g_2, \ldots, g_k \rangle \) such that \( g_i G_j = g_j g_i \). When encoding 1 qubit, \( X_L, Y_L, Z_L \) are given by cosets of \( G \) in \( G^\perp \). If \( \text{wt}(G^\perp) > t \) we want \( X_L, Y_L, Z_L \) with \( \text{wt}(t+1) \) in each of 3 cosets \( G^\perp/G \).