Homework Corrections: Problem Set 2
- Problem 2: \( S = \langle \mathbb{Z} \mathbb{I}, \mathbb{I} \mathbb{Z} \rangle \)
  \( \{ \mathbf{1}, \mathbf{Z} \mathbf{X}, \mathbf{Y} \mathbf{Z} \mathbf{I} \} \)

Today: Toric Codes
- 9-qubit quantum code
  \( |0_L\rangle = \frac{1}{\sqrt{3}} (|000\rangle + |111\rangle) |\mathbb{Z}\rangle^3 \)
  \( |1_L\rangle = \frac{1}{\sqrt{2}} (|000\rangle - |111\rangle) |\mathbb{Z}\rangle^3 \)

Suppose error \( E^{(1)} \), then state is \( \frac{1}{\sqrt{3}} (|000\rangle - |111\rangle) \langle 100| + \langle 111| \langle 111| \) \( E^{(2)} \), then state is """"

\( \Rightarrow \) this code is degenerate

* Toric Code: special kind of CSS code
  Consider grid with qubits on each edge.

\[ \text{plaqutes} \]

\[ \text{in stabilizer group} \]

\[ \text{commute} \]

\[ \Rightarrow \text{TORIC (CSS) CODE} \]

What is dimension? 16 group generators (constraints)
\( \Rightarrow 16 + 24 = 40 \)
24 group generators for plaquettes
24 group generators for vertices

\( \Rightarrow \) This code encodes \( \mathbb{C} \)-clim subspace.

* Toric code: identify the boundaries like a torus.
  There are now 16 \( \mathbb{Z} \)-constraints and 16 \( \mathbb{X} \)-constraints
  and 32 edges \( \Rightarrow \) encode 2 qubits

* What Pauli products commute with \( \frac{1}{\sqrt{2}} \mathbf{Z} \mathbf{X} \) and \( \frac{1}{\sqrt{2}} \mathbf{X} \mathbf{Z} \) and are not generated by these?
  Put \( \mathbf{Z} \)'s on edges so commute with \( \mathbf{X} \) at vertices
Euler's theorem => can look at simple cycles of Z's.

For example:

\[ \begin{array}{c}
\text{is this generated by}
\end{array} \]
\[ \begin{array}{c}
\frac{z_1}{z_2^n} ? \quad \text{yes}
\end{array} \]

yes, because we can decompose the simple cycle into squares.
However, this is not the case for cycles across the torus.

For example:

\[ \begin{array}{c}
\text{or}
\end{array} \]
\[ \begin{array}{c}
\frac{z_1^* \times z_2^*}{z_1^n \times z_2^n} = \frac{z_1^* \times z_2^*}{z_1^n \times z_2^n}
\end{array} \]

and \[ \begin{array}{c}
\text{can be decomposed in terms of } *
\end{array} \]

Logical qubit operations

\[ \begin{array}{c}
\frac{14}{4} \frac{q_1}{q_2} \frac{c}{c} = 14 \frac{q_1}{q_2} \frac{c}{c} = 2 \frac{q_1}{q_2} \frac{c}{c}
\end{array} \]

How do we represent X operations?

The following operators commute with X's:

\[ \begin{array}{c}
\frac{X_1^n}{X_2^n} \text{ and } \frac{X_1^n}{X_2^n}
\end{array} \]

Toyic Code

2k^2 qubits

smallest distance k encodes two qubits

\[ \begin{array}{c}
\text{This is a } [2, 2k^2, 2, k] \text{ quantum code.}
\end{array} \]

How do we correct errors?

Measure the \[ \begin{array}{c}
\text{errors:}
\end{array} \]

we want to make the points satisfy \[ \begin{array}{c}
\text{x}
\end{array} \]

Consider \[ \begin{array}{c}
\frac{z_1}{z_2^n} \frac{14}{14} = -14
\end{array} \]

then multiply with \[ \frac{z_1}{z_2^n} \] on one of four edges:

\[ \begin{array}{c}
\frac{z_1}{z_2^n} \frac{14}{14} = \frac{14}{14} \frac{14}{14} = \frac{14}{14} 14
\end{array} \]
Smallest correction is set of paths joining all X error vertices: degenerate

- Hamiltonian
  
  Energy: \[ \sum_{v} \frac{1}{2} (1 - X_{v+} X_{v} X_{v-} X_{v+}) + \sum_{l} \frac{1}{2} (1 - Z_{\text{top}} Z_{l_{\text{left}}} Z_{l_{\text{right}}}) \]

  Energy gives the number of violated constraints:

  Lowest Energy 0: codeword

  Second Energy 2: because violations must be even. quasi particles with energy 1.

  Suppose we had: what happens when:

  (move X particle around Z)

  First consider the sequence of operations

  That is, \[ X_1 X_{v-1} X_{v} X_{v+} Z_{l_{\text{left}}} Z_{l_{\text{right}}} X_{v-1} X_{v} X_{v+} Z_{l_{\text{left}}} Z_{l_{\text{right}}} \]

  \( V_i V_j \) intersect in exactly one spot

  \[ \Rightarrow X_1 Z_{l_{\text{left}}} Z_{l_{\text{left}}} X_1 \{ 14 \} = -14 \]

  This is equivalent to the X particle moving around Z particle. Then \[ 14 \Rightarrow -14 \]

  People call X and Z magnetic and electric charges or anyons

  - What would happen if we moved X around a torus

   Thus, would apply a logical \( T_2 \) to encoded state. Because \[ \text{is equivalent to} \]

   \[ \text{is equivalent to} \]
Quantum Codes on Qudits
\{i0, 1\} \{11, 12\}

What are analogs of X, Y, Z?

\[ T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & \omega^2 \\ \omega & \omega \end{pmatrix}, \quad \omega = -\frac{1}{2} + \frac{\sqrt{13}}{2}i = 1^{13} \]

\[ RT = \begin{pmatrix} \omega & \omega^2 \\ \omega^2 & \omega \end{pmatrix} = \omega TR \]

Qudit codes: Instead of X, Y, Z, find leasor products of \(RT^0\) which all commute.

Find quantum subspace \( \{9, 14\} \subset \{14\} \forall \).

- What are the generators for a toric code?

\[ R \begin{pmatrix} T_k \end{pmatrix} \]

\[ R \begin{pmatrix} T_k \end{pmatrix} = R_k T_i T_j \]

\[ T_i T_j = T_j T_i R_i R_j \]

\[ \omega^3 \]

\[ \omega \]

\[ \omega^2 \]

\[ \omega \]

get \( \omega \) or \( \omega^2 \) depending on clock or counter-clockwise.

Anyons because you can get any phase.

These are Abelian anyons.

Non-Abelian anyons are created from non-Abelian groups. In this case, \(j\) applies a unitary operation to the encoded subspace (instead of only a phase for Abelian anyons).

For sufficiently complicated non-Abelian anyons give universal quantum computation.
What are the elementary operations that anyons can create out of vacuum?

1. Particle/antiparticle creation
2. Move around each other (braiding)
3. Fuse two anyons
   See what type of particle you get

These operations, maybe together with classically controlled operations, give universal quantum computation.

Additionally, anyons are naturally fault-tolerant if you keep anyons far apart.