Problem 1: Coupled representation (Solved Problem)

A nucleus consists of two spin 1/2 nucleons, \( s_1 = \frac{1}{2} \) and, \( s_2 = \frac{1}{2} \). Both nucleons are in the orbital angular momentum \( l = 0 \).

a) How many spin states are there for each nucleon?

Solution:
Each nucleon can be in two states \( |\uparrow\rangle \) and \( |\downarrow\rangle \), since \(-s \leq m_s \leq s\).

b) How many spin states does the system have (based on the uncoupled representation)?

Solution:
In the uncoupled representation good quantum numbers correspond to the eigenvalues of the operators \( \hat{S}_1^2 \), \( \hat{S}_2^2 \), \( \hat{S}_{1,z} \), \( \hat{S}_{2,z} \). Since \( s_1, s_2 = \frac{1}{2} \) while \( m_s \) for each particle can take two values, we can list four possible states: \( |\uparrow\uparrow\rangle \), \( |\uparrow\downarrow\rangle \), \( |\downarrow\uparrow\rangle \), \( |\downarrow\downarrow\rangle \).

c) Which quantum numbers would you use to label the coupled representation states?

Solution:
In the coupled representation a complete set of commuting observables is given by \( \hat{S}^2 \), \( \hat{S}_2 \), \( \hat{S}_{1,z} \), \( \hat{S}_{2,z} \), thus good quantum numbers are \( s, m_s, s_1, s_2 \) (in this case, we can omit \( s_1 \) and \( s_2 \) since they’re always \( \frac{1}{2} \) and write a state as \( |s, m_s\rangle \).)

Problem 2: Commutation of angular momentum

a) (Solved question) Prove the commutation relation,

\[
[\hat{L}^2_1, \hat{L}^2_2] = 0
\]

and

\[
[\hat{L}_{1,z}, \hat{L}^2_2] \neq 0
\]

where \( \hat{L} = \hat{L}_1 + \hat{L}_2 \).

Solution:
From the definition of \( \hat{L} \), we have that the norm of the total angular momentum is \( \hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + 2\hat{L}_1 \cdot \hat{L}_2 \). Also, operators acting on different particles always commute, e.g., \( [\hat{L}^2_1, \hat{L}_2] = 0 \), \( [\hat{L}_{1,z}, \hat{L}_{2,y}] = 0 \), since they are functions of different variables.

Then the first commutator is very easily evaluated:

\[
[\hat{L}^2_1, \hat{L}^2_2] = [\hat{L}^2_1, \hat{L}^2_2 + 2\hat{L}_1 \cdot \hat{L}_2] = 0
\]

For the second commutator, we use the fact that \( [\hat{L}^2_2, \hat{L}_{1,z}] = 0 \) to simplify the result:

\[
[\hat{L}_{1,z}, \hat{L}^2_2] = [\hat{L}_{1,z}, \hat{L}_1^2 + \hat{L}_2^2 + 2\hat{L}_1 \cdot \hat{L}_2] = [\hat{L}_{1,z}, \hat{L}_1^2 + 2\hat{L}_1 \cdot \hat{L}_2] = 2 [\hat{L}_{1,z}, \hat{L}_1 \hat{L}_2] = 2 [\hat{L}_{1,z}, \hat{L}_1 \hat{L}_2] + 2 [\hat{L}_{1,z}, \hat{L}_1 y \hat{L}_2 y]
\]
We finally also use the formula \([A, BC] = B[A, C] + [A, B]C\) to find the final result:

\[
\left[\hat{L}_{1,z}, \hat{L}^2\right] = 2 \left[\hat{L}_{1,z}, \hat{L}_{1,x}\right] \hat{L}_{2x} + 2 \left[\hat{L}_{1,z}, \hat{L}_{1,y}\right] \hat{L}_{2y} = 2i\hbar(\hat{L}_{1y}\hat{L}_{2x} - \hat{L}_{1x}\hat{L}_{2y})
\]

b) Prove the commutation relation,

\[
\left[\hat{L}_x, \hat{L}^2\right] = 0
\]

and discuss why the same relation holds for the other, \((y, z)\), components of the angular momentum.

**Problem 3: Angular momentum operator**

Suppose a system is in the angular momentum state \(|7, 4\rangle\), with \(l = 7\) and \(m_x = 4\).

a) What are the possible measurement results for the \(x\) component of angular momentum?

b) What are the possible measurement values for the \(y\) component of the angular momentum?

c) What are the possible measurement results for the \(z\) component of angular momentum?

d) What is \(\Delta L_y = \sqrt{\langle L_y^2 \rangle - \langle L_y \rangle^2}\) for the state \(|7, 4\rangle\) if we assume \(\Delta L_y = \Delta L_z\)?

**Problem 4: Ladder operators**

Consider a system in the state \(|l, m_z\rangle\), that is, in an eigenstate of the \(L_z\) angular momentum with eigenvalue \(\hbar m_z\) and with total angular momentum quantum number is \(l\) [i.e. the state is also an eigenstate of \(L^2\) with eigenvalue \(\hbar^2(l + 1)\)].

a) Consider the **ladder operators** \(\hat{L}_+ = \hat{L}_x + i\hat{L}_y\) and \(\hat{L}_- = \hat{L}_x - i\hat{L}_y\). What is \(\hat{L}_\pm |l, m_z\rangle\) (see lecture notes and Griffiths)?

What is then the expectation value of \(\hat{L}_\pm\) for the state considered \((|l, m_z\rangle)\)?

b) Using the result you found above, prove that the result you found in **Problem 3:c** (the value of \(\langle L_z \rangle\)) is in general true for any eigenstate of \(L_z, L^2\).

**Problem 5: Sum of angular momenta**

The electron in an hydrogen atom is in the state \(\psi(r, \vartheta, \varphi) = R_{21}(r) \left( \frac{1}{\sqrt{3}} Y_1^0(\vartheta, \varphi)|\downarrow\rangle + \frac{\sqrt{2}}{3} Y_1^{-1}(\vartheta, \varphi)|\uparrow\rangle \right)\), where \(|\uparrow\rangle = |m_s = \frac{1}{2}\rangle\) and \(|\downarrow\rangle = |m_s = -\frac{1}{2}\rangle\) are eigenstates of the intrinsic spin with eigenvalues \(+\frac{\hbar}{2}\) and \(-\frac{\hbar}{2}\) respectively, \(Y_l^m(\vartheta, \varphi) = [l, m]\) are eigenfunctions of \(L^2\) and \(L_z\), with quantum numbers \(l\) and \(m\) and \(R_{21}(r)\) is the radial part of the wavefunction.

a) **(Solved question)** Using the sum rules, find the possible values of the quantum number \(j\) (which sets the eigenvalue of \(\hat{J}^2\) to \(\hbar^2 j(j + 1)\)), where \(\hat{J} = \hat{L} + \hat{S}\) is the total angular momentum.

**Solution:**

The sum rules state that in the addition of two angular momentum operators, we have that \(|l_1 - l_2| \leq l \leq l_1 + l_2\). In this case we have:

\[
l - s \leq j \leq l + s \quad \rightarrow \quad 1 - \frac{1}{2} \leq j \leq 1 + \frac{1}{2} \quad \rightarrow \quad \frac{1}{2} \leq j \leq \frac{3}{2},
\]

Since the angular momentum quantum number can only increase by integers between the minimum and the maximum value, we have that there are only two possible values for \(j\), \(j = \frac{1}{2}\) and \(j = \frac{3}{2}\).

b) The wavefunction

\[
\psi(r, \vartheta, \varphi) = R_{21}(r) \left( \frac{1}{\sqrt{3}} Y_1^0(\vartheta, \varphi)|\downarrow\rangle + \frac{\sqrt{2}}{3} Y_1^{-1}(\vartheta, \varphi)|\uparrow\rangle \right)
\]
can also be written as:

\[ \psi(r, \theta, \varphi) = R_{21}(r) \left( \frac{2}{3} \sqrt{2} \left| j = \frac{3}{2}, m_j = -\frac{1}{2}, s = \frac{1}{2}, l = 1 \right> - \frac{1}{3} \left| j = \frac{1}{2}, m_j = -\frac{1}{2}, s = \frac{1}{2}, l = 1 \right> \right) \]

Which one of these two expressions is the coupled representation? Is the second expression consistent with what was found in the previous question? How would you find the second expression for \( \psi \) from the first one?

c) What are the possible outcomes and probabilities of a measurement of \( L_z^2, L_z, S_z, J^2 \) and \( J_z \)?

d) (Solved question) Two \( p \) electrons \((l_1 = l_2 = 1)\) are in a state with angular momentum \(|l, m, l_1, l_2\rangle = |2, -1, 1, 1\rangle\). What are the possible values of \( m_{1z} \) and \( m_{2z} \)?

Solution:
From the state \(|2, -1, 1, 1\rangle\) we know that \( l_1 = 1 \) and \( l_2 = 1 \). Thus \( m_{1z} = \{-1, 0, 1\} \) and \( m_{2z} = \{-1, 0, 1\} \). The values of \( m_{1z} \) and \( m_{2z} \) must add up to give \( m = -1 \). We can obtain this result in two ways: either \( m_{1z} = -1 \) and \( m_{2z} = 0 \) or vice-versa, \( m_{1z} = 0 \) and \( m_{2z} = -1 \). Thus notice that the eigenvalues of \( L_{1z} \) and \( L_{2z} \) are not known from the coupled representation state: indeed these two operators do not commute with the total angular momentum \( L^2 \) so in general we cannot know the eigenvalue of \( L^2 \) and of \( L_{1z} \) and \( L_{2z} \) with certainty at the same time.

e) Two \( p \) electrons \((l_1 = l_2 = 1)\) are in a state with angular momentum \(|l, m, l_1, l_2\rangle = |2, -2, 1, 1\rangle\). What are the possible outcomes of a measurement of \( L_z^1 \)? What are the probabilities of each of these outcomes? What is the joint probability of measuring for both electrons \( L_z^1 = L_z^2 = -h \)?
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