Problem 1: Short Questions 24 points

These short questions require only short answers (but even for yes/no questions give a brief explanation)

1) What information about a quantum system can you obtain from the wavefunction?

2) If we measure the kinetic energy of a quantum particle and immediately after we measure its momentum, is the result of the second measurement random?

3) What does the Coulomb term in the Semi-empirical mass formula describe?

4) A particle is in the quantum state, \( \psi(y) = A e^{-iy}. \)
   a) What are the possible results of a momentum measurement?
   b) What are the probabilities of each possible momentum measurement?
   c) What physical situation is represented by this quantum state?

5) When is the wavefunction describing a quantum system an energy eigenfunction?

6) Which one of the following statements (if any) is correct, based on the properties of the angular momentum and its eigenfunctions?
   a) A particle is in the angular momentum eigenstate, \( \psi_{l,m}(\vartheta,\phi) = |l=3, m_z=-4.\)
   b) A particle is in the angular momentum eigenstate, \( \psi_{l,m_x,m_z}(\vartheta,\phi) = |l=4, m_x=3, m_z=2.\)
   c) A particle is in the angular momentum eigenstate, \( \psi_{l,m_x}(\vartheta,\phi) = |l=4, m_z=3.\)

7) When is a quantum system “bound”? Give a condition in terms of the system energy \( E \) and potential energy \( V.\)

8) Is the Q-value of a nuclear reaction (such as alpha-decay) the only factor that determines if the reaction does happen spontaneously?

Problem 2: Rotations and angular momentum 26 points

a) Consider classical rotations in a 3D Euclidean space. We define \( R_{\vec{n}}(\vartheta) \) the operator describing a rotation around the axis \( \vec{n} \) by an angle \( \vartheta.\) Do the operators \( R_z(\vartheta) \) and \( R_z(\varphi) \) commute? Do the operators \( R_x(\vartheta) \) and \( R_y(\varphi) \) commute? (a yes/no answer is enough)

b) Now we consider rotations in quantum mechanics. We write rotations as the operators \( \hat{R}_{\vec{n}}(\vartheta).\) For small angles \( \vartheta \) we can write these rotations using the angular momentum operator as \( \hat{R}_{\vec{n}}(\vartheta) = 1 - i\frac{\vartheta}{\hbar}\hat{L}_{\vec{n}} \) (for example \( \hat{R}_x(\vartheta) = 1 - i\frac{\vartheta}{\hbar}\hat{L}_x \)). Do rotations in quantum mechanics commute?

c) Calculate the difference between making first a rotation \( R_y(\varphi) \) followed by a rotation \( R_x(\vartheta) \) and making first \( R_x(\vartheta) \) and then \( R_y(\varphi).\) Can you express this difference as a rotation?

d) A quantum system is in a state \( \psi \) such that it is left unchanged by a rotation along \( x: \hat{R}_x(\vartheta)\psi = \psi.\) Is \( \psi \) an eigenfunction of \( \hat{L}_x?\)

e) We studied in class that the eigenvalues of the angular momentum operator along \( x, \hat{L}_x, \) are \( \hbar m_x \) with integers \( m_x = -l, -l+1, \ldots, l.\) Consider a quantum state \( \psi = \frac{1}{\sqrt{6}}\varphi_{-2} + \frac{1}{2}\varphi_0 + \frac{\sqrt{3}}{2}\varphi_1, \) where \( \varphi_m \) is the normalized eigenfunction of \( \hat{L}_x \) corresponding to the eigenvalue \( \hbar m_x.\)
What is the probability of finding \( \hat{L}_x = 0 \) in a measurement? What is \( \langle \hat{L}_x \rangle?\)
Problem 3: Radioactive decay by proton emission  30 points

Useful quantities: Proton mass, $m_p c^2 = 938.272 \text{ MeV}$; $\hbar c = 197 \text{MeV fm}$; $\frac{\hbar c^2}{\hbar c} = \frac{1}{137}$; $c = 3 \times 10^8 \text{m/s}$; $R_0 = 1.2 \text{fm}$.

a) Consider the isotope Europium-131 ($^{131}_{63}\text{Eu}$), with mass 121919.966 MeV. Given its $A$ and $Z$ numbers, do you expect this isotope to be stable?

A possible decay channel for $^{131}_{63}\text{Eu}$ is proton emission. We want to analyze this decay mode following the same theory we saw for alpha decay and in particular estimate the half-life of $^{131}_{63}\text{Eu}$. The following questions will guide you through the estimation.

b) The mass of Samarium-130 is 120980.755 MeV. What is the Q-value for the reaction $^{131}_{63}\text{Eu} \rightarrow ^{130}_{62}\text{Sa} + ^1_1\text{H}$?

c) Calculate the frequency $f = \frac{Q}{R}$ for the proton to be at the edge of the Coulomb potential. Here $R$ is the Samarium radius and $v$ the proton speed when taking $Q$ as the (classical) kinetic energy.

d) What is the Coulomb potential at the distance $R$, $V_C(R)$? (this is the potential barrier height). What is the distance $R_c$ at which the Coulomb potential is equal to the Q-value?

e) To estimate the tunneling probability we replace the Coulomb barrier with a rectangular barrier of height $V_H = V_C(R)/2$ and length $L = (R_c - R)/2$ (see figure). What is the tunneling probability?

f) Finally, give the decay rate $\lambda$ and the half-life for the proton emission decay of $^{131}_{63}\text{Eu}$.

Problem 4: Match the potential  20 points

A quantum system in a 1D geometry is subjected to the potential energy as in the figures on the right, with 5 regions of different potential height.

Match the 1D energy eigenfunctions on the left with the correct energy (if any) depicted on the right. Provide a brief explanation of the reasoning that lead you to each of your matchings.

(Notice: here I plot the real part of the eigenfunction).