The Fourier transform is a generalization of the complex Fourier series. The complex Fourier Series is an expansion of a periodic function (periodic in the interval \([-L/2, L/2]\)) in terms of an infinite sum of complex exponential:

\[
\sum_{n=-\infty}^{\infty} A_n e^{2\pi i nx/L} \tag{1}
\]

where the coefficients \(A_n\) are:

\[
A_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-2\pi i nx/L} dx. \tag{2}
\]

Note that this expansion of a periodic function is equivalent to using the exponential functions \(u_n(x) = e^{2\pi i nx/L}\) as a basis for the function vector space of periodic functions. The coefficient of each “vector” in the basis are given by the coefficient \(A_n\). Accordingly, we can interpret equation (2) as the inner product \(\langle u_n(x) | f(x) \rangle\).

In the limit as \(L \to \infty\) the sum over \(n\) becomes an integral. The discrete coefficients \(A_n\) are replaced by the continuous function \(F(k)\) where \(k = n/L\). Then in the limit \((L \to \infty)\) the equations defining the Fourier series become:

\[
f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi i kx} dk \tag{3}
\]

\[
F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i kx} dx. \tag{4}
\]

Here, \(F(k) = F_x[f(x)](k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i kx} dx\) is called the forward Fourier transform, and

\[
f(x) = F^{-1}_k[F(k)](x) = \int_{-\infty}^{\infty} F(k) e^{2\pi i kx} dk
\]

is called the inverse Fourier transform. The notation \(F_x[f(x)](k)\) is common but \(\hat{f}(k)\) and \(\tilde{f}(x)\) are sometimes also used to denote the Fourier transform.

In physics we often write the transform in terms of angular frequency \(\omega = 2\pi \nu\) instead of the oscillation frequency \(\nu\) (thus for example we replace \(2\pi k \to k\)). To maintain the symmetry between the forward and inverse transforms, we will then adopt the convention

\[
F(k) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx
\]

\[
f(x) = F^{-1}[F(k)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk.
\]

### Sine-Cosine Fourier Transform

Since any function can be split up into even and odd portions \(E(x)\) and \(O(x)\),

\[
f(x) = \frac{1}{2} [f(x) + f(-x)] + \frac{1}{2} [f(x) - f(-x)] = E(x) + O(x),
\]

a Fourier transform can always be expressed in terms of the Fourier cosine transform and Fourier sine transform as

\[
F_x[f(x)](k) = \int_{-\infty}^{\infty} E(x) \cos(2\pi kx) dx - i \int_{-\infty}^{\infty} O(x) \sin(2\pi kx) dx.
\]
Properties of the Fourier Transform

- The smoother a function (i.e., the larger the number of continuous derivatives), the more compact its Fourier transform.
- The Fourier transform is linear, since if \( f(x) \) and \( g(x) \) have Fourier transforms \( F(k) \) and \( G(k) \), then
  \[
  \int [af(x) + bg(x)]e^{-2\pi i kx} \, dx = a \int f(x)e^{-2\pi i kx} \, dx + b \int g(x)e^{-2\pi i kx} \, dx = aF(k) + bG(k).
  \]

Therefore,

\[
F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)] = aF(k) + bG(k).
\]

- The Fourier transform is also symmetric since \( F(k) = F_x[f(x)](k) \) implies \( F(-k) = F_x[f(-x)](k) \).
- The Fourier transform of a derivative \( f'(x) \) of a function \( f(x) \) is simply related to
  \[
  \int [af(x) + bg(x)]e^{-2\pi i kx} \, dx = a \int f(x)e^{-2\pi i kx} \, dx + b \int g(x)e^{-2\pi i kx} \, dx = aF(k) + bG(k).
  \]

Therefore,

\[
F[f'(x)](k) = \int f(x)e^{-2\pi i kx} \, dx.
\]

Now use integration by parts

\[
\int v \, du = [uv] - \int u \, dv
\]

with

\[
 du = f'(x) \, dx \quad v = e^{-2\pi i kx}
\]

and

\[
 u = f(x) \quad dv = -2\pi i ke^{-2\pi i kx} \, dx,
\]

then

\[
F_x[f'(x)](k) = \left[ f(x)e^{-2\pi i kx} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \left( -2\pi i ke^{-2\pi i kx} \right) \, dx
\]

The first term consists of an oscillating function times \( f(x) \). But if the function is bounded so that

\[
\lim_{x \to \pm\infty} f(x) = 0
\]

(as any physically significant signal must be), then the term vanishes, leaving

\[
F_x[f'(x)](k) = 2\pi ik \int_{-\infty}^{\infty} f(x)e^{-2\pi i kx} \, dx = 2\pi ikF_x[f(x)](k).
\]

This process can be iterated for the \( n^{th} \) derivative to yield

\[
F_x[f^{(n)}(x)](k) = (2\pi i k)^n F_x[f(x)](k).
\]

- If \( f(x) \) has the Fourier transform \( F_x[f(x)](k) = F(k) \), then the Fourier transform has the shift property
  \[
  \int_{-\infty}^{\infty} f(x-x_0)e^{-2\pi i kx} \, dx = \int_{-\infty}^{\infty} f(x-x_0)e^{-2\pi i (x-x_0)}e^{-2\pi i k x_0} \, dx = e^{-2\pi i k x_0}F(k),
  \]
  so \( f(x-x_0) \) has the Fourier transform
  \[
  F_x[f(x-x_0)](k) = e^{-2\pi i k x_0}F(k).
  \]

- If \( f(x) \) has a Fourier transform \( F_x[f(x)](k) = F(k) \), then the Fourier transform obeys a similarity theorem.
  \[
  \int_{-\infty}^{\infty} f(ax)e^{-2\pi i kx} \, dx = \frac{1}{|a|} \int_{-\infty}^{\infty} f(\frac{x}{a})e^{-2\pi i (\frac{k}{a})x} \, dx = \frac{1}{|a|}F(k/a),
  \]
  so \( f(ax) \) has the Fourier transform
  \[
  F_x[f(ax)](k) = |a|^{-1}F(k/a).
  \]

- Any operation on \( f(x) \) which leaves its area unchanged leaves \( F(0) \) unchanged, since
  \[
  \int_{-\infty}^{\infty} f(x) \, dx = F_x[f(x)](0) = F(0).
  \]
Table of common Fourier transform pairs.

<table>
<thead>
<tr>
<th>Function</th>
<th>$f(x)$</th>
<th>$F(k) = F_x<a href="k">f(x)</a>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>1</td>
<td>$\delta(k)$</td>
</tr>
<tr>
<td>Delta function</td>
<td>$\delta(x - x_0)$</td>
<td>$e^{-2\pi ikx_0}$</td>
</tr>
<tr>
<td>Cosine</td>
<td>$\cos(2\pi k_0 x)$</td>
<td>$\frac{1}{2} [\delta(k - k_0) + \delta(k + k_0)]$</td>
</tr>
<tr>
<td>Sine</td>
<td>$\sin(2\pi k_0 x)$</td>
<td>$\frac{1}{2} [\delta(k + k_0) - \delta(k - k_0)]$</td>
</tr>
<tr>
<td>Exponential function</td>
<td>$e^{-2\pi k_0</td>
<td>x</td>
</tr>
<tr>
<td>Gaussian</td>
<td>$e^{-ax^2}$</td>
<td>$\sqrt{\frac{\pi}{a}} e^{-\pi k^2 / a}$</td>
</tr>
<tr>
<td>Heaviside step function</td>
<td>$H(x)$</td>
<td>$\frac{1}{2} [\delta(k) - i/(\pi k)]$</td>
</tr>
<tr>
<td>Lorentzian function</td>
<td>$\frac{1}{\pi} \frac{\Gamma/2}{(x - x_0)^2 + (\Gamma/2)^2}$</td>
<td>$e^{-2\pi ikx_0} e^{-\pi</td>
</tr>
</tbody>
</table>

In two dimensions, the Fourier transform becomes

$$F(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(k_x, k_y) e^{-2\pi i (k_x x + k_y y)} dk_x dk_y f(k_x, k_y)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) e^{2\pi i (k_x x + k_y y)} dx dy.$$  

Similarly, the n-dimensional Fourier transform can be defined for $k, x$ in $\mathbb{R}^n$ by

$$F(x) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f(k) e^{-2\pi i k \cdot x} d^n k f(k) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} F(x) e^{2\pi i k \cdot x} d^n x.$$