Equations

• Slice Selection: \( \text{ideal } O_{2D}(x,y) = \int_{\Omega} O_{3D}(x,y,z) \delta(z - z_s)dz \)

• Better Approximation: \( O_{2D}(x,y) = \int_{\Omega} O_{3D}(x,y,z) \text{TopHat} \left( \frac{z - z_s}{1/2 \Delta z} \right) dz \)

• Radon Transform: \( P(\theta,z) = \int \int O(x,y) \delta(x \cos(\theta) + y \sin(\theta) + z) \)

• Fourier Transform: \( \hat{g}(k) = \int G(x)e^{-ikx} dx \)

• Central Slice Theorem: \( k_z = k_x \cos(\theta) + k_y \sin(\theta) \)
\( z = x \cos(\theta) + y \sin(\theta) \)

• Back Projection: \( O_r(x,y) = \frac{1}{\pi} \int_0^\pi P(\theta,z) d\theta \)

• Map: \( \hat{O}_r(k_x,k_y) = \hat{P}(\theta,k_z) \)

where \( k_z = k_x \cos(\theta) + k_y \sin(\theta) \)
The PSF associated with the simple Bach projection is:

\[ PSF|_{BF} = \frac{1}{r} \]

\[ \therefore O_r(x, y) = O(x, y) \otimes \frac{1}{\sqrt{x^2 + y^2}} \]

where \( O_r(x, y) = B\{P(\alpha, z)\} \)

and \( B = \frac{1}{\pi} \int_0^\infty P(\alpha_1 x \cos(\alpha) + y \sin(\alpha)) d\alpha \)

\[ \therefore \quad O_r(x, y) = O(x, y) \otimes \frac{1}{\sqrt{x^2 + y^2}} \]

\[ \approx = \approx k \quad O_r(k_x, k_y) = \approx k \quad O(k_x, k_y) \cdot \frac{1}{|k|} \]

so

\[ \approx O(k_x, k_y) = |k| \approx O_r(k_x, k_y) \]
More Organized Proof of The Central Slice Theorem

1. \[ P(\alpha,z) = \iint_{Ox} O(x,y) \delta(x \cos(\alpha) + y \sin(\alpha) - z) \, dx \, dy \]

2. Equate the z–axis with a tilted reference frame
   \[ x' \parallel z, \quad y' \perp z \]
   \[ \therefore x = x' \cos(\alpha) - y' \sin(\alpha) \]
   \[ y = x' \sin(\alpha) + y' \cos(\alpha) \]
   and
   \[ x' = x \cos(\alpha) + y \sin(\alpha) \]

3. Substitute #2 into #1 and change integral to \( dx' \, dy' \) (still over all space)
   \[ P(\alpha,z) = \iint_{Ox} O(x' \cos(\alpha) - y' \sin(\alpha), x' \sin(\alpha) + y' \cos(\alpha)) \delta(x' - z) \, dx' \, dy' \]

4. Integrate along \( x' \) and note that \( z \) is only a point along the \( x' \) axis.
   \[ P(\alpha,x') = \iint_{Ox} O(x' \cos(\alpha) - y' \sin(\alpha), x' \sin(\alpha) + y' \cos(\alpha)) \, dy' \]

5. Fourier Transform along \( x' \)
   \[ \tilde{p}(\alpha,k_x) = \iint_{Ox} O(x' \cos(\alpha) - y' \sin(\alpha), x' \sin(\alpha) + y' \cos(\alpha)) e^{-ix \cdot k_x} \, dx' \, dy' \]
More Organized Proof of The Central Slice Theorem

6. Transform back to the \((x,y)\) coordinate system
\[
\tilde{p}(\alpha, k_x) = \int \int O(x, y) e^{-i(x \cos(\alpha) + y \sin(\alpha))k_x} \, dx \, dy
\]

7. Define the tilted \(k\)-space coordinate system.
\[
k_x = k_x \cdot \cos(\alpha) - k_y \cdot \sin(\alpha)
\]
\[
k_y = k_x \cdot \sin(\alpha) - k_y \cdot \cos(\alpha)
\]

8. Rewrite #6 as
\[
\tilde{p}(\alpha, k_x) = \int \int O(x, y) e^{-i\left(k_x \cdot \cos(\alpha) - k_y \cdot \sin(\alpha)\right)x} e^{-i\left(k_x \cdot \sin(\alpha) + k_y \cdot \cos(\alpha)\right)y} \, dx \, dy \bigg|_{k_y=0}
\]
\[
\tilde{p}(\alpha, k_x) = \int \int O(x, y) e^{-ik_x x} e^{-ik_y y} \, dx \, dy \bigg|_{k_y=0}
\]
\[
= F_{2D} \{O(x, y)\} \bigg|_{k_y=0}
\]
The Central Slice Theorem

Consider a 2-dimensional example of an emission imaging system. \( O(x,y) \) is the object function, describing the source distribution. The projection data, is the line integral along the projection direction.

\[
P(0^o,y) = \int O(x,y)dx
\]

The Central Slice Theorem can be seen as a consequence of the separability of a 2-D Fourier Transform.

\[
\hat{o}(k_x,k_y) = \int O(x,y) e^{-ik_x x} e^{-ik_y y} dxdy
\]

The 1-D Fourier Transform of the projection is,

\[
\tilde{p}(k_y) = \int P(0^o,y) e^{-ik_y y} dy
\]

\[
= \int O(x,y) e^{-ik_y y} dxdy
\]

\[
= \int O(x,y) e^{-ik_y y} e^{-i\theta x} dxdy
\]

\[
= \hat{o}(0,k_y)
\]
The Central Slice Theorem

The one-dimensional Fourier transformation of a projection obtained at an angle $J$, is the same as the radical slice taken through the two-dimensional Fourier domain of the object at the same angle.