Solution of Problem Set 1
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Problem 1 (20Pts)
(a) The current can be expressed as \( I = N Z e \). \( N \) is the rate of protons captured by the Tungsten target; \( Z \) is 1 for proton.
\[
N = \frac{I}{e} = \frac{2.0 \times 10^{-3} \text{ A}}{1.6 \times 10^{-19} \text{ C}} = 1.25 \times 10^{16} \text{ sec}^{-1}
\]
(b) The force, by using Newton 2\(^{nd}\) law, is:
\[
F = \frac{dp}{dt} = N(0 - p) = Np
\]
where \( p \) is the linear momentum of individual protons before smashed into the target.
\[
p = \frac{1}{c} \sqrt{E^2 - E_0^2} = \frac{1}{c} \sqrt{(E_0 + E_k)^2 - E_0^2} = \frac{1}{c} \sqrt{2E_0E_k - E_0^2}
\]
\[
= \frac{1}{3 \times 10^8} \times \sqrt{2 \times 1000 \times 938.28 + 1000^2 \times 1.6 \times 10^{-13}} = 9.05 \times 10^{-39} \text{ Newton} \cdot \text{sec}
\]
\[
F = -Np = -1.25 \times 10^{16} \times 9.05 \times 10^{-39} = -1.13 \times 10^{-2} \text{ Newton}
\]
The force protons exert to the target is the anti-force of the force exert to the proton by the target. So they are equal in terms of magnitude. Thus, the magnitude of the force protons exert to the target is 1.13 \times 10^{-2} \text{ Newton}.

(c) \[
\frac{E - E_0 \times 100\%}{E_0} = \frac{E_k \times 100\%}{E_0} = \frac{1000}{938.28} = 106.6\%
\]
(d) \[
\# \text{neutrons} = N \times n(\text{neutrons each proton produced}) = 1.25 \times 10^{16} \times 20 = 2.5 \times 10^{17} \text{ sec}^{-1}
\]

Problem 2 (20Pts)
We note that \( E_n = 0.025 \text{ eV} \ll m_n c^2 \). So we calculate the velocity of neutron from its kinetic energy as follows:
\[
v_n = \sqrt{\frac{2E_n}{m_n}} = \sqrt{\frac{2 \times 0.025 \times 1.6 \times 10^{-19}}{1.66 \times 10^{-37}}} = 2.195 \times 10^7 \text{ m/s} = 2.195 \times 10^5 \text{ cm/s}
\]
The flux of neutrons can be obtained as:
\[
\phi = \rho \left( \frac{\#}{\text{cm}^3} \right) \cdot \left( \frac{\text{cm}}{\text{sec}} \right) = \frac{\#}{\text{cm}^2 \text{ sec}}.
\]
And 1 cm\(^3\) of observation volume includes \( N_0 = 9.111 \times 10^3 \) neutrons. The number of neutrons decayed within 1 minute can be calculated as:
\[
N_{\text{decay}} = N_0 - N_0 e^{-\lambda t} = N_0 (1 - e^{-\frac{t}{\lambda}}) = 9111 \times (1 - e^{-\frac{60}{10000}}) = 525.4
\]

Problem 3 (20Pts)
Assume that the nucleus is static before collision and the collision is elastic.
\( M_n \) and \( M_A \) are the masses of the neutron and the nucleus; \( v_n \) and \( v_n' \), the velocity of neutron before and after collision; \( v_A \) and \( v_A' \), the velocity of nucleus before and after collision. So \( v_n' = 0 \), the maximum kinetic energy the nucleus can obtain from the collision can be computed from the following two equations:

\[
\begin{align*}
\frac{1}{2} M_n v_n^2 + 0 &= \frac{1}{2} M_n v_n'^2 + \frac{1}{2} M_A v_A'^2 \\
M_n v_n + 0 &= M_n v_n' + M_A v_A'
\end{align*}
\]

\[ \Rightarrow E_A = \frac{4M_n M_A}{(M_n + M_A)^2} E_n \]

where \( E_A \) is the maximum kinetic energy the nucleus can obtain from the collision and \( E_n \) is the kinetic energy of the neutron before collision.

- Hydrogen: \( M_H = 1 \) a.m.u. \( E_H = \frac{4M_n}{(1 + M_n)^2} E_n \)
- Nitrogen: \( M_N = 14 \) a.m.u. \( E_N = \frac{56M_n}{(14 + M_n)^2} E_n \)

\[ \Rightarrow \frac{E_H}{E_N} = \frac{1}{14} \left( \frac{14 + M_n}{1 + M_n} \right)^2 \Rightarrow M_n = 1.0655 \text{ a.m.u.} \]

In compare with the modern value of \( M_n = 1.0086650 \text{ a.m.u.} \), the error is \( \text{about 5.63\%} \).

Problem 4 (20Pts)
(a) The thermal efficiency of an ideal Carnot engine is
\[ \eta_i = 1 - \frac{T_L}{T_H} = 1 - \frac{100 + 273.15}{500 + 273.15} = 0.517 \]
So the real thermal efficiency of the system is
\[ \eta = 0.4 \times \eta_i = 20.7\% \]
Typical fission can be written as \( ^{235}U + n \rightarrow ^{93}Rb + ^{141}Cs + 2n \). So each fission even will cost \( 235 \times 1.66 \times 10^{-21} = 3.901 \times 10^{-27} \) gram \( ^{235}U \). In order to provide 1000MW thermal power, \( ^{235}U \) used up per day can be determined by

\[
\frac{1000 \times 10^6 \times 86400}{200 \times 10^6 \times 1.6 \times 10^{-19} \times 20.7\%} \times 3.901 \times 10^{-27} = 5088 \text{ gram/day}
\]

(b) The number of \( ^{235}U \) nuclei undergoing fission per second can be determined as

\[
\frac{1000 \times 10^6 \times 1}{200 \times 10^6 \times 1.6 \times 10^{-19} \times 20.7\%} = 1.51 \times 10^{20} \text{ sec}^{-1}
\]

(c) \[
\frac{1000 \times 10^6 \times 86400}{7000 \times 4.1868 \times 20.7\%} \times 10^{-6} = 1.42 \times 10^4 \text{ ton/day}
\]

Problem 5 (20Pts)
(a) Activity \( A = \lambda N \), where \( A = \) activity, \( \lambda = \) decay constant (0.693/half life), and \( N = \) number of atoms present. Also, in terms of units, activity is defined as the number of disintegrations (also known as transformations) that occur per unit time.
The initial rate of production \((s^{-1})\) of activated atoms is \(\left(\frac{dN_{\text{act}}}{dt}\right)_0 = R\). As the population of active atoms increase, they decay at the rate \(\lambda N_{\text{act}}(s^{-1})\). Thus the net rate at which they accumulate can be expressed as
\[
\frac{dN_{\text{act}}}{dt} = \left(\frac{dN_{\text{act}}}{dt}\right)_0 - \lambda N_{\text{act}} = R - \lambda N_{\text{act}} \tag{1}
\]

General solution of this differential equation after one transit \((t)\) is \(N_{\text{act}} = \frac{R}{\lambda} (x_1 + x_2 e^{-\lambda t})\) \((2)\)

Differentiating with respect to time gives \(\frac{dN_{\text{act}}}{dt} = \frac{R}{\lambda} (-x_2 \cdot \lambda \cdot e^{-\lambda t})\) \(\tag{3}\)

Now substituting Eqs. (2) and (3) into Eq. (1), we get
\[
\frac{dN_{\text{act}}}{dt} = \frac{R}{\lambda} (-x_2 \cdot \lambda \cdot e^{-\lambda t}) = R - \lambda N_{\text{act}} = R - R(x_1 + x_2 e^{-\lambda t})
\]
\[
\frac{R}{\lambda} (-x_2 \cdot \lambda \cdot e^{-\lambda t}) = R - R(x_1 + x_2 e^{-\lambda t}), N_{\text{act}} \bigg|_{t=0} = 0
\]
\[
\Rightarrow x_1 = 1, x_2 = -1
\]
\[
\therefore N_{\text{act}} = \frac{R}{\lambda} (1 - e^{-\lambda t})
\]

The activity \(A\) added per cm\(^3\) of the coolant per transit of the target is given by
\[
A = \frac{\dot{\lambda} N_{\text{act}}}{\lambda} = R \left(1 - e^{-\lambda t}\right)
\]

(b) At very first, the activity is 0. Then after one time through the target, the activity becomes \(R(1 - e^{-\lambda t})\). We can do the same as (a). We calculate as above. It is obvious that second time the coolant goes through the external circuit and the target \((t_0 + t_i)\), the activity added is \(R\left(1 - e^{-\lambda t}\right) e^{-\lambda(t_0 + t_i)}\). And so on...

So that, after \(m\) cycles, the activity per cm\(^3\) of the coolant leaving the target is
\[
R(1 - e^{-\lambda t}) + R(1 - e^{-\lambda t})e^{-\lambda(t_0 + t_i)} + R(1 - e^{-\lambda t})e^{-\lambda(t_0 + t_i)} e^{-\lambda(t_1 + t_i)} + \ldots
\]
\[
+ R(1 - e^{-\lambda t})e^{-m\lambda(t_0 + t_i)} = R \frac{(1 - e^{-\lambda t})(1 - e^{-m\lambda(t_0 + t_i)})}{1 - e^{-\lambda(t_0 + t_i)}}
\]

(c) The maximum coolant activity at the exit is:
\[
\lim_{m \to \infty} A_m = R \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda(t_0 + t_i)}}
\]