14% 1. Reduce the following ordinary differential equation to a first-order vector differential equation, which you should write out completely, in vector format.

$$
\left( \frac{d^3 y}{dx^3} \right)^2 - 2 \frac{d^2 y}{dx^2} - \frac{dy}{dx} - y^3 = 0.
$$

18% 2. Consider an approximate discrete step in $x$ and $y$, starting at $x_n, y_n$ of the ODE $dy/dx = f(y, x)$. The Taylor expansion of the derivative function along the solution in terms of $\delta x = x - x_n$ is

$$
f(y(x), x) = f_n + \frac{df_n}{dx} \delta x + \frac{d^2 f_n}{dx^2} \frac{\delta x^2}{2!} + \ldots. \quad (1)
$$

Subscript $n$ on $f$ and its derivatives denotes evaluated at $x_n, y_n$. The approximate scheme is the following for the step from $x_n$ to $x_{n+1} = x_n + \Delta x$:

“Evaluate $y^{(1)} = y_n + f_n \Delta x$, then take the step to be $y_{n+1} = y_n + f(y^{(1)}, x_n + \Delta x)$.”

Document the accuracy of this scheme, using the notation $x_n + \Delta x/2 = x_{n+\frac{1}{2}}$ as follows.

(a) Express the exact solution for $y(x)$ as a Taylor expansion.

(b) Express the quantity $y^{(1)} - y(x_n + \Delta x/2)$ in terms of the Taylor expansion.

(c) Express $f(y^{(1)}, x_{n+\frac{1}{2}}) - f(y(x_{n+\frac{1}{2}}), x_{n+\frac{1}{2}})$ to lowest order in $y^{(1)} - y(x_{n+\frac{1}{2}})$ using $\partial f/\partial y$.

(d) Hence find an expression for $y_{n+1}$ correct to third order in $\Delta x$, and state the order to which this scheme is accurate.

18% 3. A diffusion equation in 2 dimensions with suitably normalized time units is

$$
\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2},
$$
on a finite domain with fixed $\psi$ on the boundary. It is to be advanced in time using an explicit scheme.

$$
\psi_{j,k}^{(n+1)} - \psi_{j,k}^{(n)} = \Delta t \mathbf{D} \psi^{(n)}.
$$

where $\psi^{(n)}$ denotes the value at the $n$th time step. The matrix $\mathbf{D}$ represents the finite difference form of the spatial differential operator $\nabla^2$ on a uniform grid with spacing $\Delta x$ and $\Delta y$ in the $x$ and $y$ directions, whose indices are $j, k$.

(a) Write out the right-hand-side ($\Delta t \mathbf{D} \psi^{(n)}$) of the above discrete difference equation in terms of a stencil of coefficients (whose values you should specify) times values $\psi_{j,k}$ at adjacent $j, k$ positions, to complete the formulation of the difference scheme.

(b) Consider a particular Fourier mode $\propto \exp(i k_x x) \exp(i k_y y)$. Substitute it into the difference equation, and rearrange the resultant into the form $\psi^{(n+1)} = A \psi^{(n)}$. Hence find the amplification factor, $A$. 

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(c) Deduce the condition that $\Delta t$ must satisfy to make this mode stable.
(d) By deciding which $k_x$ and $k_y$ are the most unstable, deduce the requirement on $\Delta t$ for the whole scheme to be stable.

4. Consider the partial differential system in time $t$ and one spatial coordinate $x$

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f = 0$$

where in terms of the components of $u$ (which, incidentally, is not a velocity):

$$u = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad f = \begin{pmatrix} v \\ v^2/u + w \\ -kv \end{pmatrix},$$

with $k$ a constant. Use the chain rule of spatial differentiation of $f(u)$ to write the equations as

$$\frac{\partial}{\partial t} u = -J \frac{\partial}{\partial x} u.$$

(a) Find the entire $3 \times 3$ matrix $J$ and write it out in tabular form.
(b) Find the eigenvalues of $J$.
(c) Under what conditions is this system hyperbolic?
(d) Assuming these conditions are satisfied, what are the characteristic speeds of propagation of disturbances?
(e) If a suitable explicit discrete finite difference scheme is used to solve this system numerically, then it is stable provided that the Courant-Friedrichs-Lewy (CFL) condition is satisfied. Unless you have lots of unused time, don’t derive this condition for any particular scheme. Instead, just state how it relates $\Delta t$, $\Delta x$ and the characteristic speeds of propagation.

5. A random variable is required, distributed on the interval $0 \leq x \leq 1$ with probability distribution $p(x) = 2(1-x)$. A library routine is available that returns a uniform random variate (i.e. with uniform probability $0 \leq y \leq 1$). Give formulas and an algorithm to obtain the required randomly distributed $x$ value from the returned $y$ value.

6. (a) Write out Boltzmann’s equation governing the velocity distribution function $f(t, x, v)$ in time, $t$, and one-dimension in space $x$, and velocity $v$, for particles subject to a positive uniform constant acceleration $a$, which collide with a uniform background of stationary targets of density $n_2$ that do nothing but absorb the particles with a cross-section, $\sigma$, independent of velocity.
(b) Sketch in phase space $(x, v)$ the paths of the trajectories (“orbits”) of the particles.
(c) Obtain the equation of the trajectories in the form $v_0 = g(x, v)$, where $v_0$ is the velocity on the orbit at position $x = 0$, and $g(v, x)$ is a (relatively simple) function of $x$ and $v$, which you must find.
(d) Prove that

$$f(x, v) = f_0(g(x, v)) \exp(-n_2\sigma x)$$
is a solution of the steady-state ($\partial/\partial t = 0$) Boltzmann equation. The function $f_0(v_0)$ is the distribution function at $x = 0$.

(c) If $f_0(v_0) = 1/(1 + v_0^2)$ for $v_0 > 0$, then find the distribution function $f(x, v)$ at position $x > 0$ and velocity $v$ such that $v_0$ is real, in steady state.

(d) If there are no particle sources in the positive half-plane $x > 0$, what is the value of $f(x, v)$ in steady state for $x > 0$, when $v$ is such that $v_0$ is imaginary? Why?
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