Distribution functions and averages: The average of a quantity, $G(v)$, over a distribution function, $f(v)$, is defined as,

$$\langle G \rangle = \frac{\int d^3v G(v) f(v)}{\int d^3v f(v)}$$

The Maxwellian distribution, in three dimensions, is,

$$f(v) = f(v_x, v_y, v_z) = n \left( \frac{m}{2\pi T} \right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right)$$

Note that the isotropy of this distribution means that all Cartesian coordinates are equivalent; there is no preferred direction.

(a) Prove that the form of $f$ is correctly normalized, i.e. that, $\int f d^3v = n$.
(b) A specific Cartesian velocity direction: $\langle v_x \rangle$
(c) The square velocity: $\langle v^2 \rangle$, and hence the average particle energy, $\langle \frac{1}{2}mv^2 \rangle$
(d) The average speed, $\langle |v| \rangle$

2. Basic facts you need to know. Find out, write down, and memorize (to 2 significant figures) the values of the following quantities: *(You may use either SI or CGS units, but I will use CGS for lecture, so these are recommended.*)

(a) The speed of light
(b) The charge on the electron
(c) The mass of the electron, $m_e c^2$, in MeV.
(d) The mass of the proton, $m_p c^2$, in MeV.
(e) The temperature in Kelvins, equal to 1 eV.
(f) The particle density of the air you breathe $(cm^{-3})$.
(g) The density of particles in water.
(h) The ionization potential of the hydrogen atom: calculate, $E_I = \frac{1}{2} \frac{m_e \gamma^2 c^4}{\hbar^2 c^2}$.

(i) The relationship between magnetic units of Gauss and Tesla.

(j) The relationship between particle density in units of $cm^{-3}$ and $m^{-3}$.

Here are two more fundamental physics constants you might find useful:

(k) Planck’s constant in Atomic units: $\hbar \equiv 1970 \text{ eV} - \text{Å} = 1.97 \times 10^{-5} \text{ eV} - \text{cm}

(l) The fine structure constant: $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$

3. Suppose the degree of ionization of a gas discharge is governed by the Saha equation,

$$\frac{n_e n_i}{n_0} = \frac{4}{(4\pi)^{5/2}} \left( \frac{m_e c^2 e^2}{\hbar c} \right)^3 \left( \frac{T}{E_I} \right)^{3/2} \exp \left( \frac{-E_I}{T} \right)$$

and the Debye length is small relative to the discharge size. Calculate approximately the temperature at which the gas is 50% ionized if, $E_I = 13.6 \text{ eV}$, and its total pressure is equal to one atmosphere.

4. Consider a plasma in which both electrons and (singly-charged) ions adopt thermal distributions with Boltzmann factors governed by the respective temperatures, $T_e$, and, $T_i$, which are different, in general. Show that a point charge, $q$, immersed in this plasma gives rise to a potential as a function of distance, $r$, from the charge:

$$\phi = \frac{q}{r} \exp \left( -r/\lambda \right)$$

in the approximation, $e\phi \ll T_e, T_i$. Obtain an expression for $\lambda$. Does the situation of cold ions, $T_i \ll T_e$, correspond to the idealized case of “immobile” ions sometimes referred to in textbooks?

5. Consider a charged sphere of radius, $a$, located far from all other objects in a plasma that has immobile ions and mobile electrons of temperature, $T_e$. The electrons can be assumed to adopt a Boltzmann distribution, $n = n_0 \exp(e\phi/T_e)$, where, $\phi$, is the electrostatic potential and, $n_0$, is the background density of the singly charged ions.

(a) Calculate the potential distribution in the plasma when the potential on the sphere is, $\phi_s$, in the approximation, $e\phi_s \ll T_e$.

(b) Sketch the form of, $\phi$, as a function of radius, $r$, in the two cases, $\lambda_D \ll a$, and, $a \ll \lambda_D$

(c) Calculate the charge on the sphere, and hence its capacitance in the presence of the plasma. (The charge can be determined from the surface electric field).

(d) Evaluate the capacitances when, $a = 10 \text{ cm}$, for the case, $T_e = 1 \text{ KeV}$, and (i) $n_0 = 10^{14} \text{ cm}^{-3}$, and (ii) $n_0 = 10^6 \text{ cm}^{-3}$, and compare with the capacitance in vacuum.