Ballooning mode equation

\[ \delta W = \frac{\pi}{\mu_0} \int d\psi W(\psi) \]

\[ W(\psi) = \int_{-\infty}^{\infty} d\chi \left[ \left( k_n^2 + k_t^2 \right) \left( \frac{1}{JB} \frac{\partial X}{\partial \chi} \right)^2 - \frac{2\mu_0 RB_p}{B^2} \frac{dp}{d\psi} \left( k_n^2 - k_t k_n^2 \right) \right] \]

Mercier Criterion

1. In using the quasimode representation we have had to assume the solution \( X_\psi \) converges sufficiently rapidly as \( \chi \to \pm \infty \).

2. Whether or not convergence is acceptable depends upon equilibrium profiles and parameters.

3. Analysis of Euler-Lagrange equation for \( X \) indicates that there are two classes of solutions for large \( |\chi| \) depending on profiles

4. Oscillating solutions give rise to unbounded energy: \( W \to \infty \). Strong convergence gives rise to bounded energy

5. Oscillatory case implies that ballooning mode formation is not valid as \( \chi \to |\chi| \). However, for this case a trial function of the form
leads to $\delta W < 0$ (instability)

7. For exponential solutions, one starts with the strongly converging solution as $\chi = -\infty$ and integrates to the right.

8. The condition of oscillatory solutions is known as the Mercier criterion. When the Mercier criterion is violated, the solutions oscillate for large $\chi$. The ballooning mode equation is not valid but this does not matter as the system is already unstable to interchanges.

9. When the solution’s do not oscillate the Mercier criterion is satisfied and the system is stable to interchanges, and the ballooning mode formalism is solid. In this case one integrates the equation and looks to see if there is a zero-crossing. If there is one, the system is unstable to ballooning modes.

10. Relation of Mercier-Suydam

Suydam: local behavior as $x \to 0$ near regular surface in space
Mercier: behavior as $\chi \to \infty$ in pseudo angle

Mercier is actually fourier transform of “Suydam like” analysis

$$e^{iS} : S \alpha (\psi - \psi_0) \int_{\chi_0}^\chi \frac{\partial}{\partial \psi_0} \left( \frac{JB_0}{R} \right) d\chi'$$
\[ \approx (\psi - \psi_0) \chi \frac{dq(\psi)}{d\psi} \]  
\underline{pseudo angle} 
\underline{radial localization}

\[ \approx x_k \]  
\underline{transform variable}

**Forms of the Mercier Criterion**

1. **Exact form** $D_M < 1/4$ for stability

   \[
   D_M = \frac{\mu_0 P}{q^2} \left[ 2 \left( \frac{RB_p \kappa_n}{B^2} \right) + \left( \frac{\Lambda}{B^4} \right) - \left( \frac{1}{B^2} \right) \left( \frac{\Lambda}{B^2} \right) \right] R^2 B_p^2 / J B^2
   \]

   \[
   \Lambda = F \left( \mu_0 P F - \frac{R^2 B_p^2}{J} \frac{\partial \hat{q}}{\partial \psi} \right)
   \]

   \[
   \langle Q \rangle = \int_0^{2\pi} \frac{QB_p^2}{R^2 B_p^2} Jd\chi / \int_0^{2\pi} \frac{B^2}{R^2 B_p^2} Jd\chi
   \]

2. For tokamaks, pressure is low: $\beta - \varepsilon, \varepsilon^2$. As with Suydam criterion, Mercier criterion is satisfied over most of the discharge because of low $\beta$. Only problem is near the origin.

   a. For a $\beta_p - 1$ tokamak with circular cross section, Mercier becomes

   \[
   \frac{r^2 q'^2}{q^2} + 4r\beta' \left( 1 - q^2 \right) > 0
   \]

   Near $r=0$ $q'$ is very small and we require

   \[
   q_0 > 1
   \]

   b. Near the origin for non-circular tokamaks, the criterion becomes

   \[
   1 < \alpha_0^3 \left[ 1 - \frac{4}{1 + 3\kappa^2} \left( \frac{3\kappa^2 - 1}{4\kappa^2 + 1} \left( \kappa^2 - \frac{2\delta}{\varepsilon} \right) + \frac{(\kappa - 1)^2 \beta_{p0}}{\kappa (\kappa + 1)} \right) \right]
   \]
for $\kappa = 1$, triangularity and $\beta_p$ have no effect.

for $\kappa > 1$, $\beta_p$ is destabilizing, $+\delta$ stabilizing

good Mercier: elongation and outward triangularity, moderate $\beta_p$ and $q_0 \geq 1$

c. Why do toroidal effects introduce such big changes since they are of order $\varepsilon$. Compute $\kappa_n = \mathbf{n} \cdot (\mathbf{b} \cdot \nabla \mathbf{b})$

Circle: $\kappa_n \approx -\frac{B_0^2}{r B_0^2} \varepsilon \mathbf{e}_z + \frac{B_0}{B_0} \mathbf{e}_0$

Torus: $\kappa_n \approx -\left(\frac{B_0^2}{r B_0^2} - \frac{\mathbf{e}_R}{R}\right) \varepsilon \mathbf{e}_0 + \frac{B_0}{B_0} \mathbf{e}_0$

$\approx -\frac{B_0^2}{r B_0^2} - \frac{1}{R_0} \left(1 - \frac{r}{R_0} \cos \theta + \ldots\right) \left(\mathbf{e}_r \cos \theta - \mathbf{e}_z \sin \theta\right)$

$\langle \kappa_r \rangle \approx -\frac{B_0^2}{r B_0^2} \left(1 - \frac{q^2}{2}\right)$
Ballooning Modes

1. Simple limit ballooning mode equation for $\beta_p - 1$, circular cross section plasma with gradient in $p$. $r\beta_p' - 1/\varepsilon$

2. $\chi \rightarrow \theta \quad J \rightarrow \frac{r}{B_0} \quad \kappa_n \rightarrow -\frac{\cos \theta}{R_0}$

   $R \rightarrow R_0 \quad B \rightarrow B_0$

   $B_p \rightarrow B_0 (r) \quad \frac{dp}{d\psi} \rightarrow \frac{1}{rB_0} \frac{dp}{dr}$

   and $\chi = \int_{2\phi}^{c} \frac{JB_0}{R} d\chi' = \frac{1}{R_0B_0} \left[ q^i (\theta - \theta_0) + \frac{2\mu_0 r q p^i}{R_0B_0^2} (\sin \theta - \sin \theta_0) \right]$

   $\frac{\partial}{\partial \chi} \left( \frac{\mu_0}{B^2} \frac{dp}{d\psi} F \right) = -\frac{2\mu_0 r}{R_0B_0} \frac{dp}{dr} \sin \theta$

3. This gives for the Euler-Lagrange equation

   $\frac{\partial}{\partial \theta} \left[ (1 + \Lambda^2) \frac{\partial X}{\partial \theta} \right] + \alpha [\Lambda \sin \theta + \cos \theta] X = 0$

   $\Lambda = S (\theta - \theta_0) - \alpha (\sin \theta - \sin \theta_0)$

   $S = \frac{r q^i}{q} \quad \alpha = -\frac{2\mu_0 r q p^i}{R_0B_0^2} = q^2 R_0 \beta_p'$

4. Solved numerically gives $S$ vs $\alpha$ diagram
I first region of stability (goes unstable at high $\alpha$)

II second region of stability (eventually becomes stable at high $\alpha$)

5. Region I maximum $\beta$. Set $\alpha \approx 6S$ to determine critical profile for maximum $\beta$. For $q_a \gg 1, q_0 = 1$

$$\beta_t \leq 3 \frac{e}{q_a} \text{ (circle)}$$

6. Numerical studies by Sykes, Yamazaki

$$\beta_t < 22 \frac{\kappa}{q_*} \quad q_* = \frac{2B_0 A}{\mu_0 R_0 I}$$

$$= 0.44 \frac{I_0}{aB_0}$$

For optimized profiles with elongation and outer triangularity

7. For $\kappa = 2, q_0 = 1.5, \varepsilon = 1/3 \rightarrow \beta_t \approx 10\%$

Second Stability

1. Why does such a region exist

2. Examine local shear

$$q(\psi) = \frac{1}{2\pi} \int_0^{2\pi} q(\psi, \chi) d\chi \quad \hat{q} = \frac{JB_0}{R}$$

local shear

$$\frac{r}{q} \frac{\partial \hat{q}}{\partial r} = S - \alpha \cos \theta$$

pressure driven modulation
3. Note: bad curvature occurs as $\theta = 0$ due to toroidal field. Stabilizing term due to shear $\alpha \left( \frac{r \hat{a}}{q} \right)^2$

![Graph showing the relationship between curvature and stability](image)

**External Kinks**

1. Consider surface current model. $p=$const, circular cross section

$$\delta W = \delta W_p + \delta W_s + \delta W_v$$

$$\delta W_p = \int \frac{B_1^2}{2\mu_0} dr \quad > 0$$

$$\delta W_v = \int \frac{\hat{B}_1^2}{2\mu_0} dr \quad > 0$$

$$\delta W_s = \frac{1}{2} \int_s ds \left| n \cdot \xi \right|^2 \left| \nabla \left( p + \frac{B^2}{2\mu_0} \right) \right|$$
\[
\int dS \left[ n \cdot \xi \right]^2 \left[ 2p \left( \kappa_n \right) + \frac{\left\langle B^2 \right\rangle}{2\mu_0} \left( \kappa_n - \kappa_n^\prime \right) \right] = \frac{Q + \dot{Q}}{2}
\]

- curvature term
- kink term

\[
\frac{\delta W}{2\pi \varepsilon^2 B_0^2 R_0 / \mu_0} = -\frac{1}{2\pi} \int_0^{2\pi} d\theta \left| \xi \right|^2 \left[ \left( \frac{B_n}{B_0} \right)^2 + \left( \frac{\beta_t}{\varepsilon} \right) \cos \theta \right]
\]

- toroidal curvature
- high $\beta$ ballooning effect
- kink term

2. Modes have the structure of pressure driven kinks

3. Stability diagram (sharp boundary model)

4. Full numerical studies using optimized profiles and cross-sections
\[ \beta_t = 0.14 \frac{\kappa}{q_*} = 0.028 \frac{I}{aB} \]

and \[ q_* > q_{*\text{min}} \]

Note, no dried \( q_{*\text{min}} \) for following modes.

5. For \( \kappa = 1, q_* = 1.7 \) \[ \beta_t = 0.08 \varepsilon \]

For optimized Troyon \( q_{*\text{min}} = 1.5, \kappa = 1.6 \)

\[ \beta_t = 0.15 \varepsilon \]

Set \( \varepsilon = 1/3 \rightarrow \beta_t = 5\% \)

Near the regime of reactor interest!!

**Requirements on \( \beta \)**

1. There are two basic fusion energy requirements where \( \beta \) enters

2. Ignition and Wall Loading

3. Ignition (\( T_e = T_i = T \))

\[ P_r = \frac{n^2 \sigma v}{4} Q_f = Q_f \frac{n^2 T^2 \sigma v}{4} \]

\[ \beta = \frac{4 \mu_0 n T}{B^2} \]

4. \[ P_r = Q_f \frac{\sigma v}{4} \frac{\beta^2 B^4}{T^2} \]

as \( 10 \text{ keV}, \sigma v = 10^{-22} \text{ m}^3/\text{sec} \)

5. \[ P_r = 0.5 \times 10^6 \beta^2 B^4 \text{ \( \omega/\text{m}^3 \)} \]

6. \[ P_r = P_e \]

\[ P_e = \frac{3n T}{\tau} = \frac{3 \beta B^2}{\tau} \frac{B}{4 \mu_0} = \frac{0.6 \beta B^2}{\tau} \text{ \( \omega/\text{m}^3 \)} \]

\[ \therefore 0.5 \times 10^6 \beta^2 B^4 = \frac{0.6 \beta B^2}{\tau} \]
\[
\beta \tau = \frac{1.2}{B^2}
\]

4. **Wall Loading**

a. \[ P_E = \eta P_f = \eta P_f V = \eta \cdot 5 \times 10^6 \beta^2 B^4 \cdot 2\pi R_0 \pi a^2 \]

b. \[ P_E = 4 \times 10^6 \beta^2 B^4 R_0 a^2 \approx 4 \times 10^6 \beta^2 B^4 \frac{R_0}{a} a^3 \approx 1.2 \times 10^6 \beta^2 B^4 a^3 \]

\[ P_E = 1.2 \times 10^6 \beta^2 B^4 a^3 \]

c. \[ P_f = P_W A \]

\[ P_f 2\pi R_0 a^2 = P_W 4\pi^2 aR_0 \]

\[ P_f = \frac{2P_W}{a} \]

\[ .5 \times 10^6 \beta^2 B^4 = \frac{2P_W}{a} \]

d. Eliminate \( a \) from step b.

\[ a = \frac{P^{1/3}_E}{\beta^{2/3} B^{4/3}} \left( \frac{1}{1.2 \times 10^6} \right)^{1/3} = 9 \times 10^{-3} \left( \frac{P_E}{\beta^2 B^4} \right)^{1/3} \]

\[ a = 9 \times 10^{-3} \left( \frac{P_E}{\beta^2 B^4} \right)^{1/3} \]

e. Substitute \( a \) into (C)

\[ .5 \times 10^6 \beta^2 B^4 = \frac{2P_W}{9 \times 10^{-3}} \left( \frac{\beta^2 B^4}{P_E} \right)^{1/3} \]

\[ \beta B^2 = 3 \times 10^{-3} \frac{P_W^{3/4}}{P_E^{1/4}} \]

f. For \( P_W = 4 \times 10^6 \) \( \text{W} \) and \( P_E = 10^9 \) watts

\[ \beta B^2 \leq 1.5 \]

g. For \( B = 5 \text{T} \) at \( R = R_0 \) then
5. The Troyon limit

\[ \beta < 0.03 \frac{I}{aB} \]

\[ \beta(\%) < \beta_N \frac{I_{MD}}{aB} \quad \beta_N = 3 \]

or

\[ \beta < 15\% \approx 15 \times \frac{1}{3} \times \frac{1.8}{1.5} = 6\% \]