In this example we will consider what happens if you bend your knees when you hit the ground if you are jumping from a height.

Imagine you have mass $m$ and jump from height $h$ at time $t = 0$. You hit the ground at time $t = t_1$. Then between $t = t_1$ and $t = t_2$ over interval $\Delta t = t_2 - t_1$ you bend your knees and lower your center of mass by a distance $\Delta h$. In the figure the red spot indicates your center of mass.

We will ask the questions:

a. What is the average force of the ground on your legs during the impact, in terms of $m$, $g$, $h$ and $\Delta h$?

b. What is $\Delta t$ over which the impact happens, in terms of $g$, $h$ and $\Delta h$?

c. If your mass is $m = 60$ kg, what is the maximum $h/\Delta h$ that you can sustain without breaking your tibia? We will assume that the compressive force per area necessary to break the tibia in the lower leg is about $1.6 \times 10^3$ bars (1 bar = $10^5$ Pa = $10^5$ N/m$^2$). The smallest cross-sectional area of the tibia, about 3.2 cm$^2$, is slightly above the ankle.

For this example, we’ll consider you to be a point particle located at your center of mass (we’ll see a bit later in the class why this is reasonable).
To solve part (a), first let’s consider the first time interval from $t = 0$ to $t = t_1$, during which you are falling. You can get your final velocity at the end of this interval from either 1D kinematics or the work-energy theorem. Let’s do the latter. We’ll assume no friction, so there’s no non-conservative work, so

$$\Delta KE + \Delta PE = 0$$

$$\frac{1}{2}mv_1^2 - mgh = 0$$

so

$$v_1 = \sqrt{2gh}$$

Now let’s consider the time interval from $t = t_1$ to $t = t_2$, during the impact. We can apply the work-energy theorem again,

$$\Delta KE + \Delta PE = W_{NC}$$

But this time, there is non-conservative work being done on you! There is a contact force from the floor on you, and its direction is antiparallel to your displacement. So the contact force does negative work on you, and this non-conservative work is $W_{NC} = -F_{floor} \Delta h$. (We’ll assume this force is constant over the interval).

The work-energy theorem for this interval gives:

$$-\frac{1}{2}mv_1^2 - mg\Delta h = -F_{floor} \Delta h$$

(where the “$\Delta$” refers to before and after the impact. Before, you have velocity $v_1$. After, you have come to a stop and you have velocity zero. Your change in P.E. over the interval is $-mg\Delta h$.)

Now we can plug in $\frac{1}{2}mv_1^2 = mgh$ from equation 1, and rearrange terms a little to get

$$F_{floor} = \frac{mgh + mg\Delta h}{\Delta h}$$

So that’s part (a).

Now for part (b), let’s find the impulse in order to find $\Delta t$, since impulse is average total force times time. Impulse is also change in momentum, $\Delta p$.

We know that your change in momentum over the interval is $\Delta p = p_2 - p_1 = 0 - mv_1 = -mv_1$.

$$\Delta p = F_{tot} \Delta t$$

(which is another way of stating Newton’s 2nd Law), where $F_{tot}$ is the total average force on you. There are two forces acting on you: gravity and
the force of the floor. So $F_{tot} = F_{floor} - mg$, where $F_{floor}$ is the average force of the floor that we just calculated in part (a).

So, the magnitude of $\Delta t$ is then $\frac{|\Delta p|}{|F_{floor} - mg|}$

Plugging in:

$$\Delta t = \frac{|\Delta p|}{|F_{floor} - mg|} = \frac{mv_1}{mg + mg\frac{\Delta h}{\Delta h} - mg}$$

The $m$’s cancel, and plugging in for $v_1$ and massaging a little more we get:

$$\Delta t = \sqrt{\frac{2gh}{g(h/\Delta h)}}$$

For part (c), we use our expression for $F_{floor}$ to get

$$F_{floor} = mg\left(\frac{h}{\Delta h} + 1\right)$$

The maximum force that the smallest area of the tibia can take is $1.6 \times 10^3$ bars times $10^5$ N/m$^2$ per bar times $3.2 \times 10^{-4}$ m$^2$ times 2 (for two legs) is $F_{max} = 1.0 \times 10^5$ N. If we take $F_{max} = F_{floor}$ when the tibia just breaks, and solving for $h/\Delta h$, we get $\left[(h/\Delta h)_{max} = 173\right]$

So, if you don’t bend your knees (take $\Delta h = 1$ cm), you will break your legs jumping from only 1.7 m. If you bend your knees 0.5 m, your leg bones may survive a leap from 87 m! (Please don’t try this yourself though!! This problem considers only damage to bones– in fact other tissues in your body could get damaged in a fall from a height of more than a few meters).

And if you are falling into something soft and cushiony, or into water, $\Delta h$ (and $\Delta t$) are relatively larger. Parachutists are trained to maximize time and displacement of impact when landing by crouching and rolling. And compare a dive to a belly-flop: small $\Delta h$ and $\Delta t$ during the collision ⇒ hurts more!