Chapter 4

Gauss's Law

Gauss's Law

4.1 Electric Flux

In Chapter 2 we showed that the strength of an electric field is proportional to the number of field lines per area. The number of electric field lines that penetrates a given surface is called an "electric flux," which we denote as Φ_E. The electric field can therefore be thought of as the number of lines per unit area.

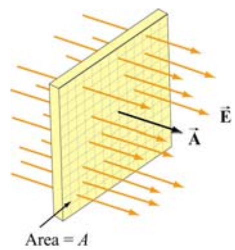


Figure 4.1.1 Electric field lines passing through a surface of area *A*.

Consider the surface shown in Figure 4.1.1. Let $\vec{\mathbf{A}} = A\hat{\mathbf{n}}$ be defined as the *area vector* having a magnitude of the area of the surface, A, and pointing in the normal direction, $\hat{\mathbf{n}}$. If the surface is placed in a uniform electric field $\vec{\mathbf{E}}$ that points in the same direction as $\hat{\mathbf{n}}$, i.e., perpendicular to the surface *A*, the flux through the surface is

$$\Phi_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}} = \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, A = EA \tag{4.1.1}$$

On the other hand, if the electric field $\vec{\mathbf{E}}$ makes an angle θ with $\hat{\mathbf{n}}$ (Figure 4.1.2), the electric flux becomes

$$\Phi_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}} = EA\cos\theta = E_{\mathrm{n}} A \tag{4.1.2}$$

where $E_{\mathrm{n}} = \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}$ is the component of $\vec{\mathbf{E}}$ perpendicular to the surface.

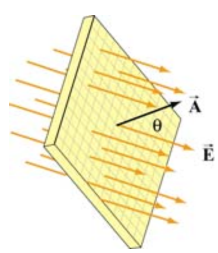


Figure 4.1.2 Electric field lines passing through a surface of area *A* whose normal makes an angle θ with the field.

Note that with the definition for the normal vector $\hat{\mathbf{n}}$, the electric flux Φ_E is positive if the electric field lines are leaving the surface, and negative if entering the surface.

In general, a surface S can be curved and the electric field $\vec{\mathbf{E}}$ may vary over the surface. We shall be interested in the case where the surface is *closed*. A closed surface is a surface which completely encloses a volume. In order to compute the electric flux, we divide the surface into a large number of infinitesimal area elements $\Delta\vec{\mathbf{A}}_i = \Delta A_i\,\hat{\mathbf{n}}_i$, as shown in Figure 4.1.3. Note that for a closed surface the unit vector $\hat{\mathbf{n}}_i$ is chosen to point in the *outward* normal direction.

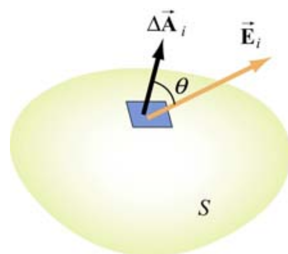


Figure 4.1.3 Electric field passing through an area element $\Delta\vec{\mathbf{A}}_i$, making an angle θ with the normal of the surface.

The electric flux through $\Delta\vec{\mathbf{A}}_i$ is

$$\Delta\Phi_E = \vec{\mathbf{E}}_i \cdot \Delta\vec{\mathbf{A}}_i = E_i \Delta A_i \cos\theta \tag{4.1.3}$$

The total flux through the entire surface can be obtained by summing over all the area elements. Taking the limit $\Delta\vec{\mathbf{A}}_i \to 0$ and the number of elements to infinity, we have

$$\Phi_E = \lim_{\Delta A_i \to 0} \sum \vec{\mathbf{E}}_i \cdot d\vec{\mathbf{A}}_i = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \tag{4.1.4}$$

where the symbol $\oiint_S$ denotes a double integral over a *closed* surface S. In order to evaluate the above integral, we must first specify the surface and then sum over the dot product $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$.

4.2 Gauss's Law

Consider a positive point charge Q located at the center of a sphere of radius r, as shown in Figure 4.2.1. The electric field due to the charge Q is $\vec{\mathbf{E}} = (Q/4\pi\varepsilon_0 r^2)\hat{\mathbf{r}}$, which points in the radial direction. We enclose the charge by an imaginary sphere of radius *r* called the "Gaussian surface."

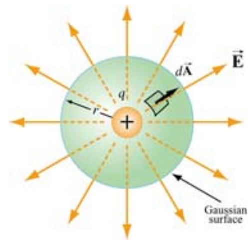


Figure 4.2.1 A spherical Gaussian surface enclosing a charge Q.

In spherical coordinates, a small surface area element on the sphere is given by (Figure 4.2.2)

$$d\vec{\mathbf{A}} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}} \tag{4.2.1}$$

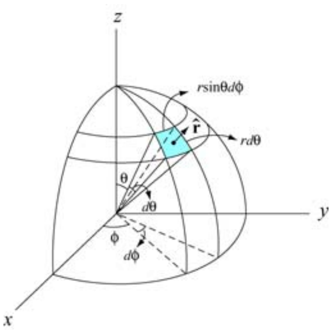


Figure 4.2.2 A small area element on the surface of a sphere of radius *r*.

Thus, the net electric flux through the area element is

$$d\Phi_E = \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E\,dA = \left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\right)\left(r^2 \sin\theta \, d\theta \, d\phi\right) = \frac{Q}{4\pi\varepsilon_0}\sin\theta \, d\theta \, d\phi \tag{4.2.2}$$

The total flux through the entire surface is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{4\pi\varepsilon_0}\int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi = \frac{Q}{\varepsilon_0} \tag{4.2.3}$$

The same result can also be obtained by noting that a sphere of radius *r* has a surface area $A = 4\pi r^2$, and since the magnitude of the electric field at any point on the spherical surface is $E = Q / 4\pi\varepsilon_0 r^2$, the electric flux through the surface is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \oiint_S dA = EA = \left(\frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \right) 4\pi r^2 = \frac{Q}{\varepsilon_0} \tag{4.2.4}$$

In the above, we have chosen a sphere to be the Gaussian surface. However, it turns out that the shape of the closed surface can be arbitrarily chosen. For the surfaces shown in Figure 4.2.3, the same result ($\Phi_E = Q/\varepsilon_0$) is obtained. whether the choice is S_1, S_2 or S_3.

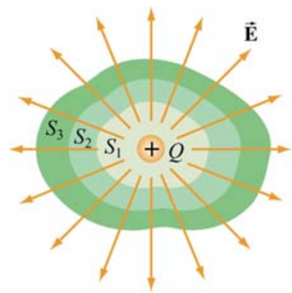


Figure 4.2.3 Different Gaussian surfaces with the same outward electric flux.

The statement that the net flux through any closed surface is proportional to the net charge enclosed is known as Gauss's law. Mathematically, Gauss's law is expressed as

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{enc}}}{\varepsilon_0} \quad \text{(Gauss's law)} \tag{4.2.5}$$

where q_{enc} is the net charge inside the surface. One way to explain why Gauss's law holds is due to note that the number of field lines that leave the charge is independent of the shape of the imaginary Gaussian surface we choose to enclose the charge.

To prove Gauss's law, we introduce the concept of the *solid angle*. Let $\Delta\vec{\mathbf{A}}_1 = \Delta A_1 \hat{\mathbf{r}}$ be an area element on the surface of a sphere S_1 of radius r_1, as shown in Figure 4.2.4.

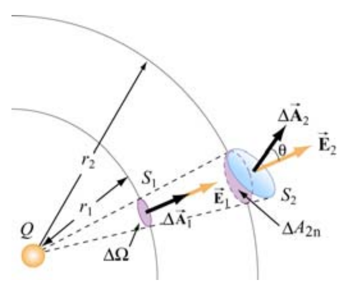


Figure 4.2.4 The area element ΔA subtends a solid angle $\Delta\Omega$.

The solid angle $\Delta\Omega$ subtended by $\Delta\vec{\mathbf{A}}_1 = \Delta A_1 \hat{\mathbf{r}}$ at the center of the sphere is defined as

$$\Delta\Omega \equiv \frac{\Delta A_1}{r_1^2} \tag{4.2.6}$$

Solid angles are dimensionless quantities measured in steradians (sr). Since the surface area of the sphere S_1 is $4\pi r_1^2$, the total solid angle subtended by the sphere is

$$\Omega = \frac{4\pi r_1^2}{r_1^2} = 4\pi \tag{4.2.7}$$

The concept of solid angle in three dimensions is analogous to the ordinary angle in two dimensions. As illustrated in Figure 4.2.5, an angle $\Delta\varphi$ is the ratio of the length of the arc to the radius r of a circle:

$$\Delta\varphi = \frac{\Delta s}{r} \tag{4.2.8}$$

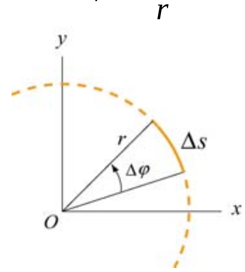


Figure 4.2.5 The arc Δs subtends an angle $\Delta\varphi$.

Since the total length of the arc is $s = 2\pi r$, the total angle subtended by the circle is

$$\varphi = \frac{2\pi r}{r} = 2\pi \tag{4.2.9}$$

In Figure 4.2.4, the area element $\Delta\vec{\mathbf{A}}_2$ makes an angle θ with the radial unit vector $\hat{\mathbf{r}}$, then the solid angle subtended by ΔA_2 is

$$\Delta\Omega = \frac{\Delta\vec{\mathbf{A}}_2 \cdot \hat{\mathbf{r}}}{r_2^2} = \frac{\Delta A_2 \cos\theta}{r_2^2} = \frac{\Delta A_{2\text{n}}}{r_2^2} \tag{4.2.10}$$

where $\Delta A_{2\text{n}} = \Delta A_2 \cos\theta$ is the area of the radial projection of ΔA_2 onto a second sphere S_2 of radius r_2, concentric with S_1.

As shown in Figure 4.2.4, the solid angle subtended is the same for both ΔA_1 and $\Delta A_{2\text{n}}$:

$$\Delta\Omega = \frac{\Delta A_1}{r_1^2} = \frac{\Delta A_2 \cos\theta}{r_2^2} \tag{4.2.11}$$

Now suppose a point charge Q is placed at the center of the concentric spheres. The electric field strengths E_1 and E_2 at the center of the area elements ΔA_1 and ΔA_2 are related by Coulomb's law:

$$E_i = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r_i^2} \quad \Rightarrow \quad \frac{E_2}{E_1} = \frac{r_1^2}{r_2^2} \tag{4.2.12}$$

The electric flux through ΔA_1 on S_1 is

$$\Delta\Phi_1 = \vec{\mathbf{E}} \cdot \Delta\vec{\mathbf{A}}_1 = E_1\,\Delta A_1 \tag{4.2.13}$$

On the other hand, the electric flux through ΔA_2 on S_2 is

$$\Delta\Phi_2 = \vec{\mathbf{E}}_2 \cdot \Delta\vec{\mathbf{A}}_2 = E_2\,\Delta A_2 \cos\theta = E_1\left(\frac{r_1^2}{r_2^2}\right)\cdot\left(\frac{r_2^2}{r_1^2}\right)A_1 = E_1\,\Delta A_1 = \Phi_1 \tag{4.2.14}$$

Thus, we see that the electric flux through any area element subtending the same solid angle is constant, independent of the shape or orientation of the surface.

In summary, Gauss's law provides a convenient tool for evaluating electric field. However, its application is limited only to systems that possess certain symmetry, namely, systems with cylindrical, planar and spherical symmetry. In the table below, we give some examples of systems in which Gauss's law is applicable for determining electric field, with the corresponding Gaussian surfaces:

Symmetry	**System**	**Gaussian Surface**	**Examples**
Cylindrical	Infinite rod	Coaxial Cylinder	Example 4.1
Planar	Infinite plane	Gaussian "Pillbox"	Example 4.2
Spherical	Sphere, Spherical shell	Concentric Sphere	Examples 4.3 & 4.4

The following steps may be useful when applying Gauss's law:

(1) Identify the symmetry associated with the charge distribution.

(2) Determine the direction of the electric field, and a "Gaussian surface" on which the magnitude of the electric field is constant over portions of the surface.

(3) Divide the space into different regions associated with the charge distribution. For each region, calculate q_{enc}, the charge enclosed by the Gaussian surface.

(4) Calculate the electric flux Φ_E through the Gaussian surface for each region.

(5) Equate Φ_E with $q_{\text{enc}} / \varepsilon_0$, and deduce the magnitude of the electric field.

Example 4.1: Infinitely Long Rod of Uniform Charge Density

An infinitely long rod of negligible radius has a uniform charge density λ. Calculate the electric field at a distance r from the wire.

Solution:

We shall solve the problem by following the steps outlined above.

(1) An infinitely long rod possesses cylindrical symmetry.

(2) The charge density is uniformly distributed throughout the length, and the electric field $\vec{\mathbf{E}}$ must be point radially away from the symmetry axis of the rod (Figure 4.2.6). The magnitude of the electric field is constant on cylindrical surfaces of radius r. Therefore, we choose a coaxial cylinder as our Gaussian surface.

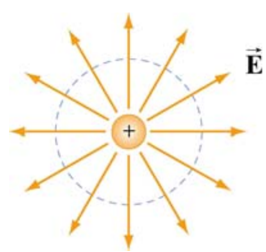


Figure 4.2.6 Field lines for an infinite uniformly charged rod (the symmetry axis of the rod and the Gaussian cylinder are perpendicular to plane of the page.)

(3) The amount of charge enclosed by the Gaussian surface, a cylinder of radius r and length ℓ (Figure 4.2.7), is $q_{\text{enc}} = \lambda\ell$.

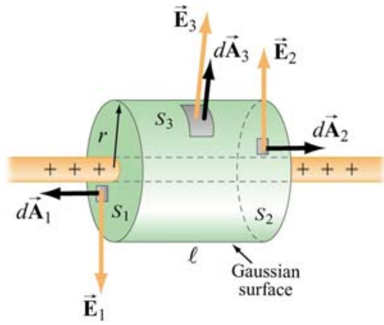


Figure 4.2.7 Gaussian surface for a uniformly charged rod.

(4) As indicated in Figure 4.2.7, the Gaussian surface consists of three parts: a two ends S_1 and S_2 plus the curved side wall S_3. The flux through the Gaussian surface is

$$\begin{aligned}\Phi_E &= \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \iint_{S_1} \vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{A}}_1 + \iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 + \iint_{S_3} \vec{\mathbf{E}}_3 \cdot d\vec{\mathbf{A}}_3 \\ &= 0 + 0 + E_3 A_3 = E(2\pi r \ell)\end{aligned} \tag{4.2.15}$$

where we have set $E_3 = E$. As can be seen from the figure, no flux passes through the ends since the area vectors $d\vec{\mathbf{A}}_1$ and $d\vec{\mathbf{A}}_2$ are perpendicular to the electric field which points in the radial direction.

(5) Applying Gauss's law gives $E(2\pi r \ell) = \lambda \ell / \varepsilon_0$, or

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \tag{4.2.16}$$

The result is in complete agreement with that obtained in Eq. (2.10.11) using Coulomb's law. Notice that the result is independent of the length ℓ of the cylinder, and only depends on the inverse of the distance r from the symmetry axis. The qualitative behavior of E as a function of r is plotted in Figure 4.2.8.

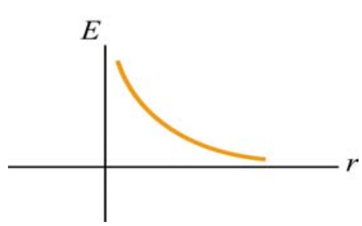


Figure 4.2.8 Electric field due to a uniformly charged rod as a function of r

Example 4.2: Infinite Plane of Charge

Consider an infinitely large non-conducting plane in the *xy*-plane with uniform surface charge density σ. Determine the electric field everywhere in space.

Solution:

(1) An infinitely large plane possesses a planar symmetry.

(2) Since the charge is uniformly distributed on the surface, the electric field $\vec{\mathbf{E}}$ must point perpendicularly away from the plane, $\vec{\mathbf{E}} = E\,\hat{\mathbf{k}}$. The magnitude of the electric field is constant on planes parallel to the non-conducting plane.

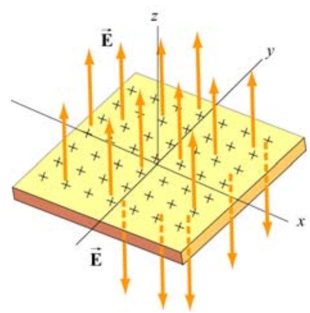


Figure 4.2.9 Electric field for uniform plane of charge

We choose our Gaussian surface to be a cylinder, which is often referred to as a "pillbox" (Figure 4.2.10). The pillbox also consists of three parts: two end-caps S_1 and S_2, and a curved side S_3.

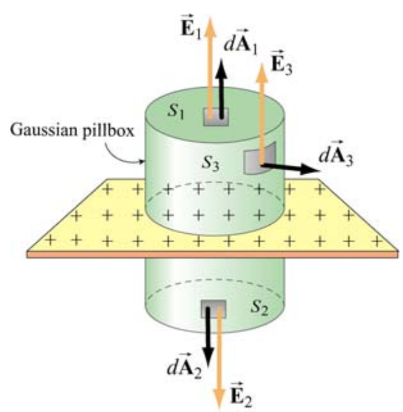


Figure 4.2.10 A Gaussian "pillbox" for calculating the electric field due to a large plane.

(3) Since the surface charge distribution on is uniform, the charge enclosed by the Gaussian "pillbox" is $q_{\text{enc}} = \sigma A$, where $A = A_1 = A_2$ is the area of the end-caps.

(4) The total flux through the Gaussian pillbox flux is

$$\begin{aligned}\Phi_E &= \oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \iint_{S_1} \vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{A}}_1 + \iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 + \iint_{S_3} \vec{\mathbf{E}}_3 \cdot d\vec{\mathbf{A}}_3 \\ &= E_1 A_1 + E_2 A_2 + 0 \\ &= (E_1 + E_2) A\end{aligned} \tag{4.2.17}$$

Since the two ends are at the same distance from the plane, by symmetry, the magnitude of the electric field must be the same: $E_1 = E_2 = E$. Hence, the total flux can be rewritten as

$$\Phi_E = 2EA \tag{4.2.18}$$

(5) By applying Gauss's law, we obtain

$$2EA = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

which gives

$$E = \frac{\sigma}{2\varepsilon_0} \tag{4.2.19}$$

In unit-vector notation, we have

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z < 0 \end{cases} \tag{4.2.20}$$

Thus, we see that the electric field due to an infinite large non-conducting plane is uniform in space. The result, plotted in Figure 4.2.11, is the same as that obtained in Eq. (2.10.21) using Coulomb's law.

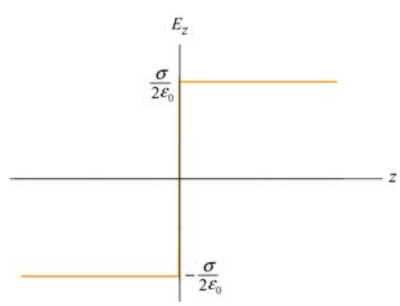


Figure 4.2.11 Electric field of an infinitely large non-conducting plane.

Note again the discontinuity in electric field as we cross the plane:

$$\Delta E_z = E_{z+} - E_{z-} = \frac{\sigma}{2\varepsilon_0} - \left(-\frac{\sigma}{2\varepsilon_0}\right) = \frac{\sigma}{\varepsilon_0} \tag{4.2.21}$$

Example 4.3: Spherical Shell

A thin spherical shell of radius a has a charge $+Q$ evenly distributed over its surface. Find the electric field both inside and outside the shell.

Solutions:

The charge distribution is spherically symmetric, with a surface charge density $\sigma = Q / A_s = Q / 4\pi a^2$, where $A_s = 4\pi a^2$ is the surface area of the sphere. The electric field $\vec{\mathbf{E}}$ must be radially symmetric and directed outward (Figure 4.2.12). We treat the regions $r \le a$ and $r \ge a$ separately.

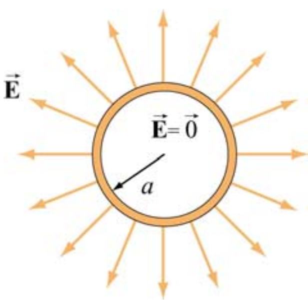


Figure 4.2.12 Electric field for uniform spherical shell of charge

Case 1: $r \le a$

We choose our Gaussian surface to be a sphere of radius $r \le a$, as shown in Figure 4.2.13(a).

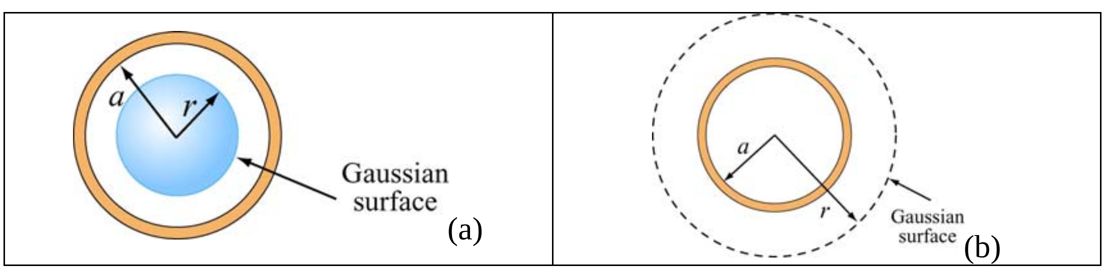


Figure 4.2.13 Gaussian surface for uniformly charged spherical shell for (a) $r < a$, and (b) $r \ge a$

The charge enclosed by the Gaussian surface is $q_{\text{enc}} = 0$ since all the charge is located on the surface of the shell. Thus, from Gauss's law, $\Phi_E = q_{\text{enc}} / \varepsilon_0$, we conclude

$$E = 0, \quad r < a \tag{4.2.22}$$

Case 2: $r \ge a$

In this case, the Gaussian surface is a sphere of radius $r \ge a$, as shown in Figure 4.2.13(b). Since the radius of the "Gaussian sphere" is greater than the radius of the spherical shell, all the charge is enclosed:

$$q_{\text{enc}} = Q$$

Since the flux through the Gaussian surface is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E(4\pi r^2)$$

by applying Gauss's law, we obtain

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} = k_e \frac{Q}{r^2}, \quad r \ge a \tag{4.2.23}$$

Note that the field outside the sphere is the same as if all the charges were concentrated at the center of the sphere. The qualitative behavior of E as a function of r is plotted in Figure 4.2.14.

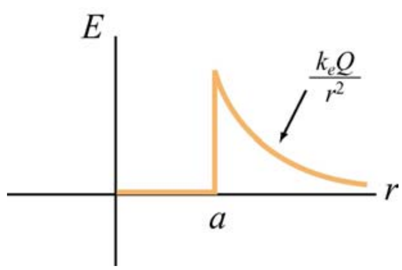


Figure 4.2.14 Electric field as a function of r due to a uniformly charged spherical shell.

As in the case of a non-conducting charged plane, we again see a discontinuity in E as we cross the boundary at $r = a$. The change, from outer to the inner surface, is given by

$$\Delta E = E_+ - E_- = \frac{Q}{4\pi\varepsilon_0 a^2} - 0 = \frac{\sigma}{\varepsilon_0}$$

Example 4.4: Non-Conducting Solid Sphere

An electric charge $+Q$ is uniformly distributed throughout a non-conducting solid sphere of radius a. Determine the electric field everywhere inside and outside the sphere.

Solution:

The charge distribution is spherically symmetric with the charge density given by

$$\rho = \frac{Q}{V} = \frac{Q}{(4/3)\pi a^3} \tag{4.2.24}$$

where V is the volume of the sphere. In this case, the electric field $\vec{\mathbf{E}}$ is radially symmetric and directed outward. The magnitude of the electric field is constant on spherical surfaces of radius r. The regions $r \le a$ and $r \ge a$ shall be studied separately.

Case 1: $r \le a$.

We choose our Gaussian surface to be a sphere of radius $r \le a$, as shown in Figure 4.2.15(a).

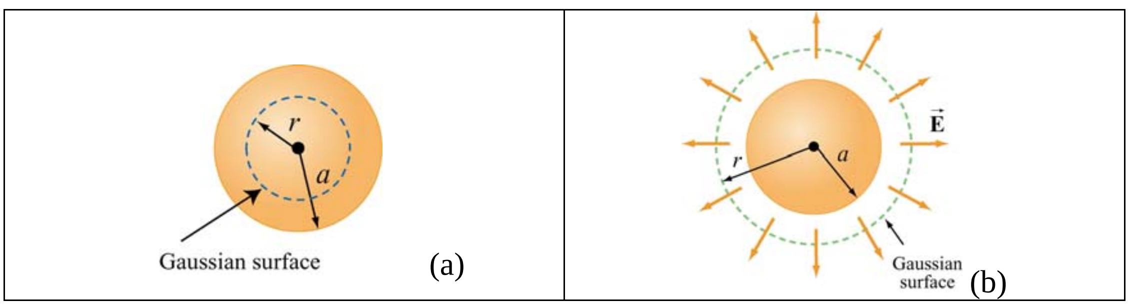


Figure 4.2.15 Gaussian surface for uniformly charged solid sphere, for (a) $r \le a$, and (b) $r > a$.

The flux through the Gaussian surface is

$$\Phi_E = \oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E(4\pi r^2)$$

With uniform charge distribution, the charge enclosed is

$$q_{\text{enc}} = \int_V \rho\, dV = \rho V = \rho\left(\frac{4}{3}\pi r^3\right) = Q\left(\frac{r^3}{a^3}\right) \tag{4.2.25}$$

which is proportional to the volume enclosed by the Gaussian surface. Applying Gauss's law $\Phi_E = q_{\text{enc}} / \varepsilon_0$, we obtain

$$E\left(4\pi r^2\right) = \frac{\rho}{\varepsilon_0}\left(\frac{4}{3}\pi r^3\right)$$

or

$$\boxed{E = \frac{\rho r}{3\varepsilon_0} = \frac{Qr}{4\pi\varepsilon_0 a^3}, \quad r \le a} \tag{4.2.26}$$

Case 2: $r \ge a$.

In this case, our Gaussian surface is a sphere of radius $r \ge a$, as shown in Figure 4.2.15(b). Since the radius of the Gaussian surface is greater than the radius of the sphere all the charge is enclosed in our Gaussian surface: $q_{\text{enc}} = Q$. With the electric flux through the Gaussian surface given by $\Phi_E = E(4\pi r^2)$, upon applying Gauss's law, we obtain $E(4\pi r^2) = Q / \varepsilon_0$, or

$$\boxed{E = \frac{Q}{4\pi\varepsilon_0 r^2} = k_e \frac{Q}{r^2}, \quad r > a} \tag{4.2.27}$$

The field outside the sphere is the same as if all the charges were concentrated at the center of the sphere. The qualitative behavior of E as a function of r is plotted in Figure 4.2.16.

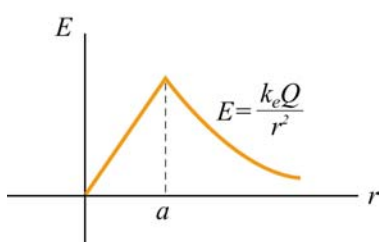


Figure 4.2.16 Electric field due to a uniformly charged sphere as a function of r.

4.3 Conductors

An insulator such as glass or paper is a material in which electrons are attached to some particular atoms and cannot move freely. On the other hand, inside a conductor, electrons are free to move around. The basic properties of a conductor are the following:

(1) The electric field is zero inside a conductor.

If we place a solid spherical conductor in a constant external field $\vec{\mathbf{E}}_0$, the positive and negative charges will move toward the polar regions of the sphere (the regions on the left and right of the sphere in Figure 4.3.1 below), thereby inducing an electric field $\vec{\mathbf{E}}'$. Inside the conductor, $\vec{\mathbf{E}}'$ points in the opposite direction of $\vec{\mathbf{E}}_0$. Since charges are mobile, they will continue to move until $\vec{\mathbf{E}}'$ completely cancels $\vec{\mathbf{E}}_0$ inside the conductor. At electrostatic equilibrium, $\vec{\mathbf{E}}$ must vanish inside a conductor. Outside the conductor, the electric field $\vec{\mathbf{E}}'$ due to the induced charge distribution corresponds to a dipole field, and the total electric field is simply $\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}'$. The field lines are depicted in Figure 4.3.1.

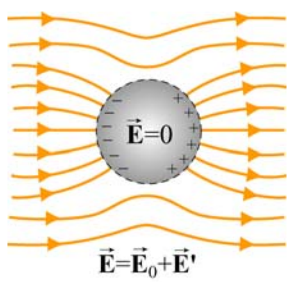


Figure 4.3.1 Placing a conductor in a uniform electric field $\vec{\mathbf{E}}_0$.

(2) Any net charge must reside on the surface.

If there were a net charge inside the conductor, then by Gauss's law (Eq. 4.3.2), $\vec{\mathbf{E}}$ would no longer be zero there. Therefore, all the net excess charge must flow to the surface of the conductor.

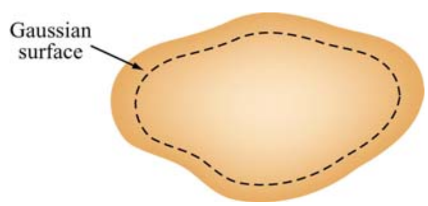


Figure 4.3.2 Gaussian surface inside a conductor. The enclosed charge is zero.

(3) The tangential component of $\vec{\mathbf{E}}$ is zero on the surface of a conductor.

We have already seen that for an isolated conductor, the electric field is zero in its interior. Any excess charge placed on the conductor must then distribute itself on the surface, as implied by Gauss's law.

Consider the line integral $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ around a closed path shown in Figure 4.3.3:

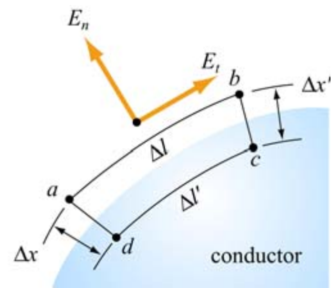


Figure 4.3.3 Normal and tangential components of electric field outside the conductor

Since the electric field $\vec{\mathbf{E}}$ is conservative, the line integral around the closed path *abcda* vanishes:

$$\oint_{abcda} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = E_t(\Delta l) - E_n(\Delta x') + 0(\Delta l') + E_n(\Delta x) = 0$$

where E_t and E_n are the tangential and the normal components of the electric field, respectively, and we have oriented the segment *ab* so that it is parallel to E_t. In the limit where both Δx and $\Delta x' \to 0$, we have $E_t \Delta l = 0$. However, since the length element Δl is finite, we conclude that the tangential component of the electric field on the surface of a conductor vanishes:

$$\boxed{E_t = 0 \ \ \text{(on the surface of a conductor)}} \tag{4.3.1}$$

This implies that the surface of a conductor in electrostatic equilibrium is an *equipotential surface*. To verify this claim, consider two points *A* and *B* on the surface of a conductor. Since the tangential component $E_t = 0$, the potential difference is

$$V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = 0$$

because $\vec{\mathbf{E}}$ is perpendicular to $d\,\vec{\mathbf{s}}$. Thus, points *A* and *B* are at the same potential with $V_A = V_B$.

(4) $\vec{\mathbf{E}}$ is normal to the surface just outside the conductor.

If the tangential component of $\vec{\mathbf{E}}$ is initially non-zero, charges will then move around until it vanishes. Hence, only the normal component survives.

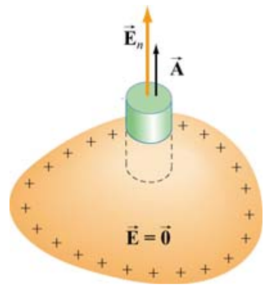


Figure 4.3.3 Gaussian "pillbox" for computing the electric field outside the conductor.

To compute the field strength just outside the conductor, consider the Gaussian pillbox drawn in Figure 4.3.3. Using Gauss's law, we obtain

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_n A + (0) \cdot A = \frac{\sigma A}{\varepsilon_0} \tag{4.3.2}$$

or

$$\boxed{E_n = \frac{\sigma}{\varepsilon_0}} \tag{4.3.3}$$

The above result holds for a conductor of arbitrary shape. The pattern of the electric field line directions for the region near a conductor is shown in Figure 4.3.4.

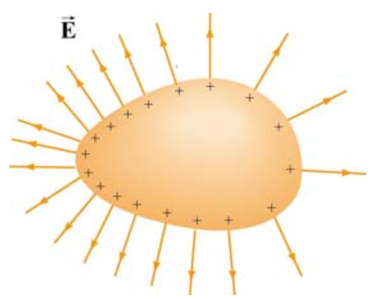


Figure 4.3.4 Just outside the conductor, $\vec{\mathbf{E}}$ is always perpendicular to the surface.

As in the examples of an infinitely large non-conducting plane and a spherical shell, the normal component of the electric field exhibits a discontinuity at the boundary:

$$\Delta E_n = E_n^{(+)} - E_n^{(-)} = \frac{\sigma}{\varepsilon_0} - 0 = \frac{\sigma}{\varepsilon_0}$$

Example 4.5: Conductor with Charge Inside a Cavity

Consider a hollow conductor shown in Figure 4.3.5 below. Suppose the net charge carried by the conductor is $+Q$. In addition, there is a charge q inside the cavity. What is the charge on the outer surface of the conductor?

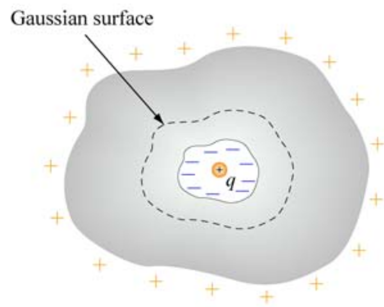


Figure 4.3.5 Conductor with a cavity

Since the electric field inside a conductor must be zero, the net charge enclosed by the Gaussian surface shown in Figure 4.3.5 must be zero. This implies that a charge $-q$ must have been induced on the cavity surface. Since the conductor itself has a charge $+Q$, the amount of charge on the outer surface of the conductor must be $Q+q$.

Example 4.6: Electric Potential Due to a Spherical Shell

Consider a metallic spherical shell of radius a and charge Q, as shown in Figure 4.3.6.

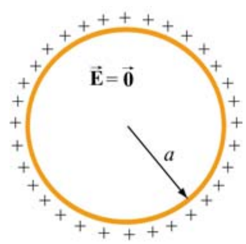


Figure 4.3.6 A spherical shell of radius a and charge Q.

(a) Find the electric potential everywhere.

(b) Calculate the potential energy of the system.

Solution:

(a) In Example 4.3, we showed that the electric field for a spherical shell of is given by

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}, & r > a \\ 0, & r < a \end{cases}$$

The electric potential may be calculated by using Eq. (3.1.9):

$$V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

For $r > a$, we have

$$V(r) - V(\infty) = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r'^2} dr' = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = k_e \frac{Q}{r} \tag{4.3.4}$$

where we have chosen $V(\infty) = 0$ as our reference point. On the other hand, for $r < a$, the potential becomes

$$\begin{aligned} V(r) - V(\infty) &= -\int_\infty^a dr E\left(r > a\right) - \int_a^r \cancel{E\left(r < a\right)} \\ &= -\int_\infty^a dr \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a} = k_e \frac{Q}{a} \end{aligned} \tag{4.3.5}$$

A plot of the electric potential is shown in Figure 4.3.7. Note that the potential V is constant inside a conductor.

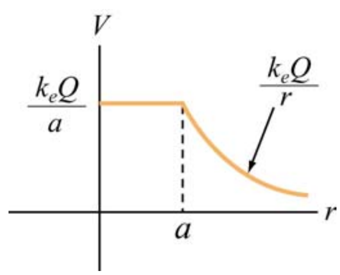


Figure 4.3.7 Electric potential as a function of r for a spherical conducting shell

(b) The potential energy U can be thought of as the work that needs to be done to build up the system. To charge up the sphere, an external agent must bring charge from infinity and deposit it onto the surface of the sphere.

Suppose the charge accumulated on the sphere at some instant is q. The potential at the surface of the sphere is then $V = q / 4\pi\varepsilon_0 a$. The amount of work that must be done by an external agent to bring charge dq from infinity and deposit it on the sphere is

$$dW_{\text{ext}} = V dq = \left(\frac{q}{4\pi\varepsilon_0 a} \right) dq \tag{4.3.6}$$

Therefore, the total amount of work needed to charge the sphere to Q is

$$W_{\text{ext}} = \int_0^Q dq \frac{q}{4\pi\varepsilon_0 a} = \frac{Q^2}{8\pi\varepsilon_0 a} \tag{4.3.7}$$

Since $V = Q / 4\pi\varepsilon_0 a$ and $W_{\text{ext}} = U$, the above expression is simplified to

$$\boxed{U = \frac{1}{2} QV} \tag{4.3.8}$$

The result can be contrasted with the case of a point charge. The work required to bring a point charge Q from infinity to a point where the electric potential due to other charges is V would be $W_{\text{ext}} = QV$. Therefore, for a point charge Q, the potential energy is $U=QV$.

Now, suppose two metal spheres with radii r_1 and r_2 are connected by a thin conducting wire, as shown in Figure 4.3.8.

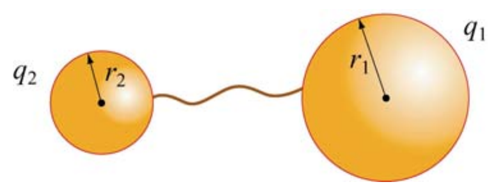


Figure 4.3.8 Two conducting spheres connected by a wire.

Charge will continue to flow until equilibrium is established such that both spheres are at the same potential $V_1 = V_2 = V$. Suppose the charges on the spheres at equilibrium are q_1 and q_2. Neglecting the effect of the wire that connects the two spheres, the equipotential condition implies

$$V = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_1} = \frac{1}{4\pi\varepsilon_0} \frac{q_2}{r_2}$$

or

$$\frac{q_1}{r_1} = \frac{q_2}{r_2} \tag{4.3.9}$$

assuming that the two spheres are very far apart so that the charge distributions on the surfaces of the conductors are uniform. The electric fields can be expressed as

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1^2} = \frac{\sigma_1}{\varepsilon_0}, \quad E_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2^2} = \frac{\sigma_2}{\varepsilon_0} \tag{4.3.10}$$

where σ_1 and σ_2 are the surface charge densities on spheres 1 and 2, respectively. The two equations can be combined to yield

$$\boxed{\frac{E_1}{E_2} = \frac{\sigma_1}{\sigma_2} = \frac{r_2}{r_1}} \tag{4.3.11}$$

With the surface charge density being inversely proportional to the radius, we conclude that the regions with the smallest radii of curvature have the greatest σ. Thus, the electric field strength on the surface of a conductor is greatest at the sharpest point. The design of a lightning rod is based on this principle.

4.4 Force on a Conductor

We have seen that at the boundary surface of a conductor with a uniform charge density σ, the tangential component of the electric field is zero, and hence, continuous, while the normal component of the electric field exhibits discontinuity, with $\Delta E_n = \sigma / \varepsilon_0$. Consider a small patch of charge on a conducting surface, as shown in Figure 4.4.1.

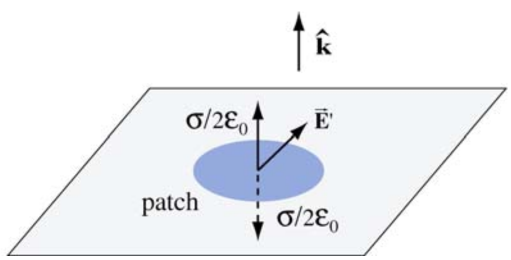


Figure 4.4.1 Force on a conductor

What is the force experienced by this patch? To answer this question, let's write the total electric field anywhere outside the surface as

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_{\text{patch}} + \vec{\mathbf{E}}' \tag{4.4.1}$$

where $\vec{\mathbf{E}}_{\text{patch}}$ is the electric field due to charge on the patch, and $\vec{\mathbf{E}}'$ is the electric field due to all other charges. Since by Newton's third law, the patch cannot exert a force on itself, the force on the patch must come solely from $\vec{\mathbf{E}}'$. Assuming the patch to be a flat surface, from Gauss's law, the electric field due to the patch is

$$\vec{\mathbf{E}}_{\text{patch}} = \begin{cases} +\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z < 0 \end{cases} \tag{4.4.2}$$

By superposition principle, the electric field above the conducting surface is

$$\vec{\mathbf{E}}_{\text{above}} = \left(\frac{\sigma}{2\varepsilon_0}\right)\hat{\mathbf{k}} + \vec{\mathbf{E}}' \tag{4.4.3}$$

Similarly, below the conducting surface, the electric field is

$$\vec{\mathbf{E}}_{\text{below}} = -\left(\frac{\sigma}{2\varepsilon_0}\right)\hat{\mathbf{k}} + \vec{\mathbf{E}}' \tag{4.4.4}$$

Notice that $\vec{\mathbf{E}}'$ is continuous across the boundary. This is due to the fact that if the patch were removed, the field in the remaining "hole" exhibits no discontinuity. Using the two equations above, we find

$$\vec{\mathbf{E}}' = \frac{1}{2}\left(\vec{\mathbf{E}}_{\text{above}} + \vec{\mathbf{E}}_{\text{below}}\right) = \vec{\mathbf{E}}_{\text{avg}} \tag{4.4.5}$$

In the case of a conductor, with $\vec{\mathbf{E}}_{\text{above}} = (\sigma/\varepsilon_0)\hat{\mathbf{k}}$ and $\vec{\mathbf{E}}_{\text{below}} = 0$, we have

$$\vec{\mathbf{E}}_{\text{avg}} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon_0}\hat{\mathbf{k}} + 0\right) = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}} \tag{4.4.6}$$

Thus, the force acting on the patch is

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}}_{\text{avg}} = (\sigma A)\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}} = \frac{\sigma^2 A}{2\varepsilon_0}\hat{\mathbf{k}} \tag{4.4.7}$$

where A is the area of the patch. This is precisely the force needed to drive the charges on the surface of a conductor to an equilibrium state where the electric field just outside the conductor takes on the value σ/ε_0 and vanishes inside. Note that irrespective of the sign of σ, the force tends to pull the patch into the field.

Using the result obtained above, we may define the electrostatic pressure on the patch as

$$\boxed{P = \frac{F}{A} = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2}\varepsilon_0\left(\frac{\sigma}{\varepsilon_0}\right)^2 = \frac{1}{2}\varepsilon_0 E^2} \tag{4.4.8}$$

where E is the magnitude of the field just above the patch. The pressure is being transmitted via the electric field.

4.5 Summary

- The **electric flux** that passes through a surface characterized by the area vector $\vec{\mathbf{A}} = A\hat{\mathbf{n}}$ is

  $$\Phi_E = \vec{\mathbf{E}}\cdot\vec{\mathbf{A}} = EA\cos\theta$$

  where θ is the angle between the electric field $\vec{\mathbf{E}}$ and the unit vector $\hat{\mathbf{n}}$.

- In general, the electric flux through a surface is

  $$\Phi_E = \iint_S \vec{\mathbf{E}}\cdot d\vec{\mathbf{A}}$$

- **Gauss's law** states that the electric flux through any closed Gaussian surface is proportional to the total charge enclosed by the surface:

  $$\Phi_E = \oiint_S \vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

  Gauss's law can be used to calculate the electric field for a system that possesses planar, cylindrical or spherical symmetry.

- The normal component of the electric field exhibits discontinuity, with $\Delta E_n = \sigma/\varepsilon_0$, when crossing a boundary with surface charge density σ.

- The basic properties of a **conductor** are (1) The electric field inside a conductor is zero; (2) any net charge must reside on the surface of the conductor; (3) the surface of a conductor is an equipotential surface, and the tangential component of the electric field on the surface is zero; and (4) just outside the conductor, the electric field is normal to the surface.

- **Electrostatic pressure** on a conducting surface is

$$P = \frac{F}{A} = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2}\varepsilon_0\left(\frac{\sigma}{\varepsilon_0}\right)^2 = \frac{1}{2}\varepsilon_0 E^2$$

4.6 Appendix: Tensions and Pressures

In Section 4.4, the pressure transmitted by the electric field on a conducting surface was derived. We now consider a more general case where a closed surface (an imaginary box) is placed in an electric field, as shown in Figure 4.6.1.

If we look at the top face of the imaginary box, there is an electric field pointing in the outward normal direction of that face. From Faraday's field theory perspective, we would say that the field on that face transmits a tension along itself across the face, thereby resulting in an *upward pull*, just as if we had attached a string under tension to that face to pull it upward. Similarly, if we look at the bottom face of the imaginary box, the field on that face is anti-parallel to the outward normal of the face, and according to Faraday's interpretation, we would again say that the field on the bottom face transmits a tension along itself, giving rise to a *downward pull*, just as if a string has been attached to that face to pull it downward. (The actual determination of the direction of the force requires an advanced treatment using the Maxwell's stress tensor.) Note that this is a *pull* parallel to the outward normal of the bottom face, regardless of whether the field is into the surface or out of the surface.

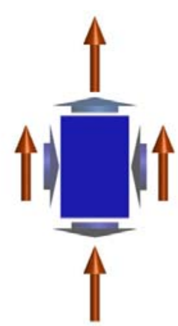


Figure 4.6.1 An imaginary box in an electric field (long orange vectors). The short vectors indicate the directions of stresses transmitted by the field, either pressures (on the left or right faces of the box) or tensions (on the top and bottom faces of the box).

For the left side of the imaginary box, the field on that face is perpendicular to the outward normal of that face, and Faraday would have said that the field on that face transmits a pressure perpendicular to itself, causing a *push* to the *right*. Similarly, for the right side of the imaginary box, the field on that face is perpendicular to the outward normal of the face, and the field would transmit a pressure perpendicular to itself. In this case, there is a *push* to the *left*.

Note that the term "tension" is used when the stress transmitted by the field is parallel (or anti-parallel) to the outward normal of the surface, and "pressure" when it is perpendicular to the outward normal. The magnitude of these pressures and tensions on the various faces of the imaginary surface in Figure 4.6.1 is given by $\varepsilon_0 E^2/2$ for the electric field. This quantity has units of force per unit area, or pressure. It is also the energy density stored in the electric field since energy per unit volume has the same units as pressure.

Animation 4.1: Charged Particle Moving in a Constant Electric Field

As an example of the stresses transmitted by electric fields, and of the interchange of energy between fields and particles, consider a positive electric charge $q > 0$ moving in a constant electric field.

Suppose the charge is initially moving upward along the positive *z*-axis in a constant background field $\vec{\mathbf{E}} = -E_0\hat{\mathbf{k}}$. Since the charge experiences a constant downward force $\vec{\mathbf{F}}_e = q\vec{\mathbf{E}} = -qE_0\hat{\mathbf{k}}$, it eventually comes to rest (say, at the origin *z* = 0), and then moves back down the negative *z*-axis. This motion and the fields that accompany it are shown in Figure 4.6.2, at two different times.

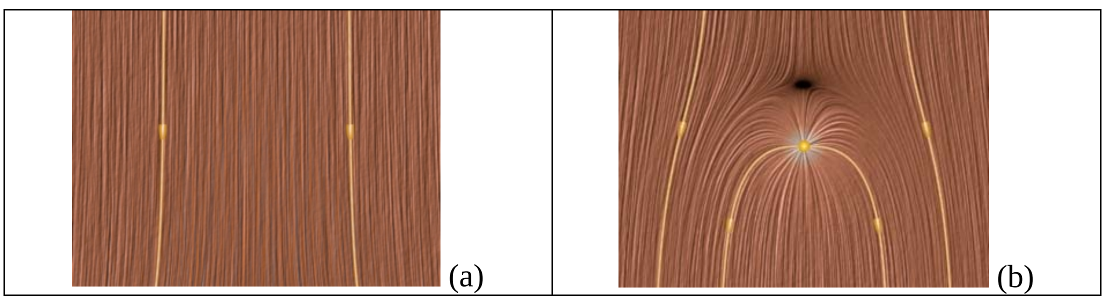


Figure 4.6.2 A positive charge moving in a constant electric field which points downward. (a) The total field configuration when the charge is still out of sight on the negative *z*-axis. (b) The total field configuration when the charge comes to rest at the origin, before it moves back down the negative *z*-axis.

How do we interpret the motion of the charge in terms of the stresses transmitted by the fields? Faraday would have described the downward force on the charge in Figure 4.6.2(b) as follows: Let the charge be surrounded by an imaginary sphere centered on it, as shown in Figure 4.6.3. The field lines piercing the lower half of the sphere transmit a tension that is parallel to the field. This is a stress pulling downward on the charge from below. The field lines draped over the top of the imaginary sphere transmit a pressure perpendicular to themselves. This is a stress pushing down on the charge from above. The total effect of these stresses is a net downward force on the charge.

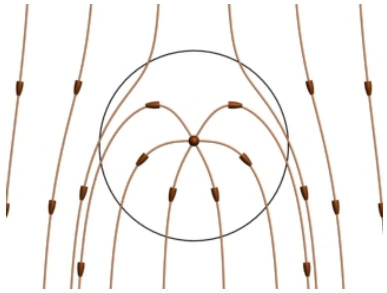


Figure 4.6.3 An electric charge in a constant downward electric field. We surround the charge by an imaginary sphere in order to discuss the stresses transmitted across the surface of that sphere by the electric field.

Viewing the animation of Figure 4.6.2 greatly enhances Faraday's interpretation of the stresses in the static image. As the charge moves upward, it is apparent in the animation that the electric field lines are generally compressed above the charge and stretched below the charge. This field configuration enables the transmission of a downward force to the moving charge we can see as well as an upward force to the charges that produce the constant field, which we cannot see. The overall appearance of the upward motion of the charge through the electric field is that of a point being forced into a resisting medium, with stresses arising in that medium as a result of that encroachment.

The kinetic energy of the upwardly moving charge is decreasing as more and more energy is stored in the compressed electrostatic field, and conversely when the charge is moving downward. Moreover, because the field line motion in the animation is in the direction of the energy flow, we can explicitly see the electromagnetic energy flow away from the charge into the surrounding field when the charge is slowing. Conversely, we see the electromagnetic energy flow back to the charge from the surrounding field when the charge is being accelerated back down the z-axis by the energy released from the field.

Finally, consider momentum conservation. The moving charge in the animation of Figure 4.6.2 completely reverses its direction of motion over the course of the animation. How do we conserve momentum in this process? Momentum is conserved because momentum in the positive z-direction is transmitted from the moving charge to the charges that are generating the constant downward electric field (not shown). This is obvious from the field configuration shown in Figure 4.6.3. The field stress, which pushes downward on the charge, is accompanied by a stress pushing upward on the charges generating the constant field.

Animation 4.2: Charged Particle at Rest in a Time-Varying Field

As a second example of the stresses transmitted by electric fields, consider a positive point charge sitting at rest at the origin in an external field which is constant in space but varies in time. This external field is uniform varies according to the equation

$$\vec{\mathbf{E}} = -E_0 \sin^4\left(\frac{2\pi t}{T}\right)\hat{\mathbf{k}} \tag{4.6.1}$$

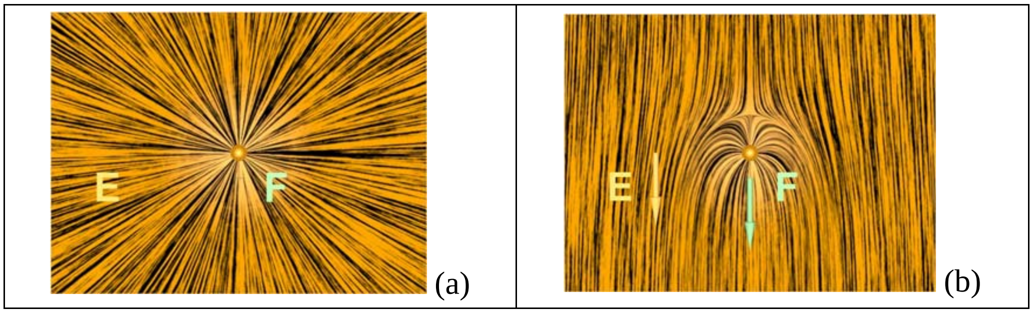


Figure 4.6.4 Two frames of an animation of the electric field around a positive charge sitting at rest in a time-changing electric field that points downward. The orange vector is the electric field and the white vector is the force on the point charge.

Figure 4.6.4 shows two frames of an animation of the total electric field configuration for this situation. Figure 4.6.4(a) is at $t = 0$, when the vertical electric field is zero. Frame 4.6.4(b) is at a quarter period later, when the downward electric field is at a maximum. As in Figure 4.6.3 above, we interpret the field configuration in Figure 4.6.4(b) as indicating a net downward force on the stationary charge. The animation of Figure 4.6.4 shows dramatically the inflow of energy into the neighborhood of the charge as the external electric field grows in time, with a resulting build-up of stress that transmits a downward force to the positive charge.

We can estimate the magnitude of the force on the charge in Figure 4.6.4(b) as follows. At the time shown in Figure 4.6.4(b), the distance r_0 above the charge at which the electric field of the charge is equal and opposite to the constant electric field is determined by the equation

$$E_0 = \frac{q}{4\pi\,\varepsilon_0\, r_0^{\,2}} \tag{4.6.2}$$

The surface area of a sphere of this radius is $A = 4\pi\, r_0^{\,2} = q/\varepsilon_0 E_0$. Now according to Eq. (4.4.8) the pressure (force per unit area) and/or tension transmitted across the surface of this sphere surrounding the charge is of the order of $\varepsilon_0 E^2/2$. Since the electric field on the surface of the sphere is of order E_0, the total force transmitted by the field is of order $\varepsilon_0 E_0^{\,2}/2$ times the area of the sphere, or $(\varepsilon_0 E_0^{\,2}/2)(4\pi r_0^2) = (\varepsilon_0 E_0^{\,2}/2)(q/\varepsilon_0 E_0) \approx qE_0$, as we expect.

Of course this net force is a combination of a pressure pushing down on the top of the sphere and a tension pulling down across the bottom of the sphere. However, the rough estimate that we have just made demonstrates that the pressures and tensions transmitted across the surface of this sphere surrounding the charge are plausibly of order $\varepsilon_0 E^2 / 2$, as we claimed in Eq. (4.4.8).

Animation 4.3: Like and Unlike Charges Hanging from Pendulums
Consider two charges hanging from pendulums whose supports can be moved closer or further apart by an external agent. First, suppose the charges both have the same sign, and therefore repel.

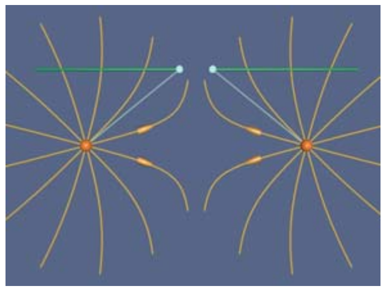


Figure 4.6.5 Two pendulums from which are suspended charges of the same sign.

Figure 4.6.5 shows the situation when an external agent tries to move the supports (from which the two positive charges are suspended) together. The force of gravity is pulling the charges down, and the force of electrostatic repulsion is pushing them apart on the radial line joining them. The behavior of the electric fields in this situation is an example of an electrostatic pressure transmitted perpendicular to the field. That pressure tries to keep the two charges apart in this situation, as the external agent controlling the pendulum supports tries to move them together. When we move the supports together the charges are pushed apart by the pressure transmitted perpendicular to the electric field. We artificially terminate the field lines at a fixed distance from the charges to avoid visual confusion.

In contrast, suppose the charges are of opposite signs, and therefore attract. Figure 4.6.6 shows the situation when an external agent moves the supports (from which the two positive charges are suspended) together. The force of gravity is pulling the charges down, and the force of electrostatic attraction is pulling them together on the radial line joining them. The behavior of the electric fields in this situation is an example of the tension transmitted parallel to the field. That tension tries to pull the two unlike charges together in this situation.

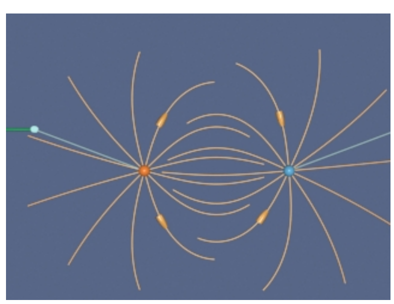


Figure 4.6.6 Two pendulums with suspended charges of opposite sign.

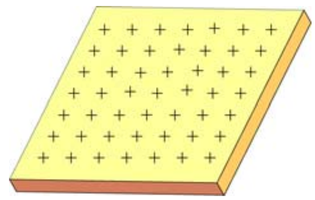
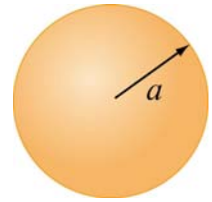
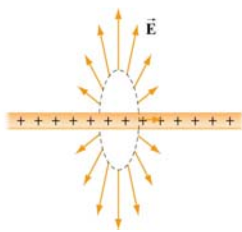
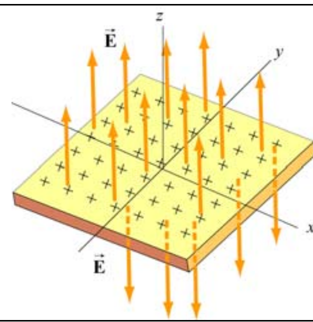
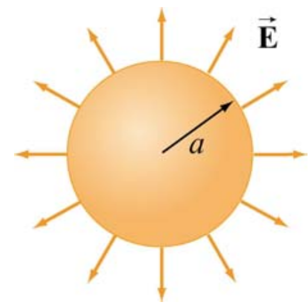
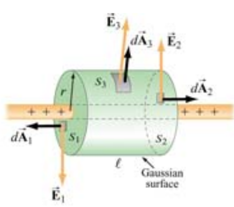
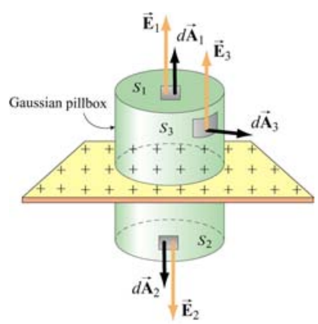
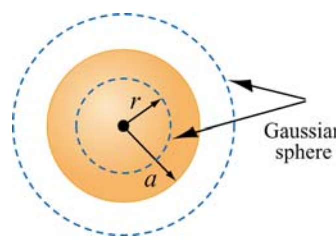
When we move the supports together the charges are pulled together by the tension transmitted parallel to the electric field. We artificially terminate the field lines at a fixed distance from the charges to avoid visual confusion.

4.7 Problem-Solving Strategies

In this chapter, we have shown how electric field can be computed using Gauss's law:

$$\boxed{\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{enc}}}{\varepsilon_0}}$$

The procedures are outlined in Section 4.2. Below we summarize how the above procedures can be employed to compute the electric field for a line of charge, an infinite plane of charge and a uniformly charged solid sphere.

System	Infinite line of charge	Infinite plane of charge	Uniformly charged solid sphere
Figure			
Identify the symmetry	Cylindrical	Planar	Spherical
Determine the direction of $\vec{\mathbf{E}}$			
Divide the space into different regions	$r > 0$	$z > 0$ and $z < 0$	$r \le a$ and $r \ge a$
Choose Gaussian surface	 Coaxial cylinder	 Gaussian pillbox	 Concentric sphere
Calculate electric flux	$\Phi_E = E(2\pi r l)$	$\Phi_E = EA + EA = 2EA$	$\Phi_E = E(4\pi r^2)$
Calculate enclosed charge q_{in}	$q_{\text{enc}} = \lambda l$	$q_{\text{enc}} = \sigma A$	$q_{\text{enc}} = \begin{cases} Q(r/a)^3 & r \le a \\ Q & r \ge a \end{cases}$
Apply Gauss's law $\Phi_E = q_{\text{in}} / \varepsilon_0$ to find E	$E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$	$E = \dfrac{\sigma}{2\varepsilon_0}$	$E = \begin{cases} \dfrac{Qr}{4\pi\varepsilon_0 a^3}, & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r \ge a \end{cases}$

4.8 Solved Problems

4.8.1 Two Parallel Infinite Non-Conducting Planes

Two parallel infinite non-conducting planes lying in the *xy*-plane are separated by a distance d . Each plane is uniformly charged with equal but opposite surface charge densities, as shown in Figure 4.8.1. Find the electric field everywhere in space.

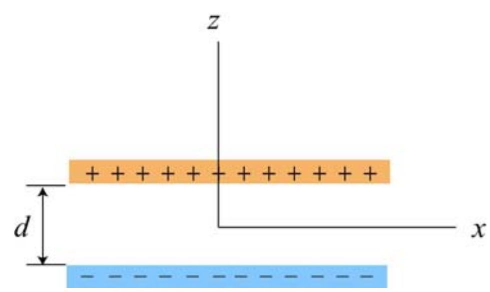


Figure 4.8.1 Positive and negative uniformly charged infinite planes

Solution:

The electric field due to the two planes can be found by applying the superposition principle to the result obtained in Example 4.2 for one plane. Since the planes carry equal but opposite surface charge densities, both fields have equal magnitude:

$$E_+ = E_- = \frac{\sigma}{2\varepsilon_0}$$

The field of the positive plane points away from the positive plane and the field of the negative plane points towards the negative plane (Figure 4.8.2)

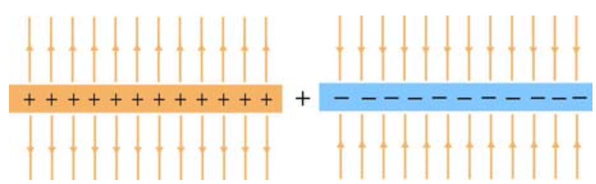


Figure 4.8.2 Electric field of positive and negative planes

Therefore, when we add these fields together, we see that the field outside the parallel planes is zero, and the field between the planes has twice the magnitude of the field of either plane.

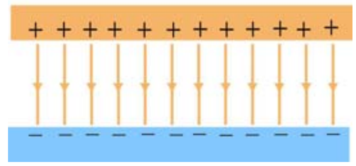


Figure 4.8.3 Electric field of two parallel planes

The electric field of the positive and the negative planes are given by

$$\vec{\mathbf{E}}_+ = \begin{cases} +\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z > d/2 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z < d/2 \end{cases} \quad , \quad \vec{\mathbf{E}}_- = \begin{cases} -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z > -d/2 \\ +\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z < -d/2 \end{cases}$$

Adding these two fields together then yields

$$\vec{\mathbf{E}} = \begin{cases} 0\,\hat{\mathbf{k}}, & z > d/2 \\ -\dfrac{\sigma}{\varepsilon_0}\hat{\mathbf{k}}, & d/2 > z > -d/2 \\ 0\,\hat{\mathbf{k}}, & z < -d/2 \end{cases} \tag{4.8.1}$$

Note that the magnitude of the electric field between the plates is $E = \sigma/\varepsilon_0$, which is twice that of a single plate, and vanishes in the regions $z > d/2$ and $z < -d/2$.

4.8.2 Electric Flux Through a Square Surface

(a) Compute the electric flux through a square surface of edges $2l$ due to a charge $+Q$ located at a perpendicular distance l from the center of the square, as shown in Figure 4.8.4.

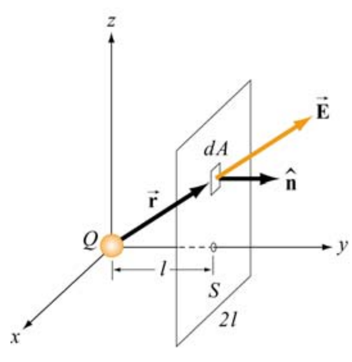


Figure 4.8.4 Electric flux through a square surface

(b) Using the result obtained in (a), if the charge $+Q$ is now at the center of a cube of side $2l$ (Figure 4.8.5), what is the total flux emerging from all the six faces of the closed surface?

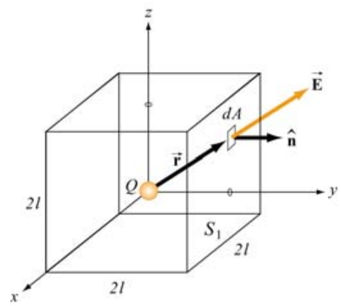


Figure 4.8.5 Electric flux through the surface of a cube

Solutions:

(a) The electric field due to the charge $+Q$ is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\left(\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{r}\right)$$

where $r=(x^2+y^2+z^2)^{1/2}$ in Cartesian coordinates. On the surface S, $y=l$ and the area element is $d\vec{\mathbf{A}} = dA\hat{\mathbf{j}} = (dx\,dz)\hat{\mathbf{j}}$. Since $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0$ and $\hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = 1$, we have

$$\vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{r}\right)\cdot(dx\,dz)\hat{\mathbf{j}} = \frac{Ql}{4\pi\varepsilon_0 r^3}dx\,dz$$

Thus, the electric flux through S is

$$\Phi_E = \oiint_S \vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = \frac{Ql}{4\pi\varepsilon_0}\int_{-l}^{l}dx\int_{-l}^{l}\frac{dz}{(x^2+l^2+z^2)^{3/2}} = \frac{Ql}{4\pi\varepsilon_0}\int_{-l}^{l}dx\left.\frac{z}{(x^2+l^2)(x^2+l^2+z^2)^{1/2}}\right|_{-l}^{l}$$

$$= \frac{Ql}{2\pi\varepsilon_0}\int_{-l}^{l}\frac{l\,dx}{(x^2+l^2)(x^2+2l^2)^{1/2}} = \frac{Q}{2\pi\varepsilon_0}\tan^{-1}\left.\left(\frac{x}{\sqrt{x^2+2l^2}}\right)\right|_{-l}^{l}$$

$$= \frac{Q}{2\pi\varepsilon_0}\left[\tan^{-1}(1/\sqrt{3}) - \tan^{-1}(-1/\sqrt{3})\right] = \frac{Q}{6\varepsilon_0}$$

where the following integrals have been used:

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}$$

$$\int \frac{dx}{(x^2+a^2)(x^2+b^2)^{1/2}} = \frac{1}{a(b^2-a^2)^{1/2}} \tan^{-1}\sqrt{\frac{b^2-a^2}{a^2(x^2+b^2)}}, \quad b^2 > a^2$$

(b) From symmetry arguments, the flux through each face must be the same. Thus, the total flux through the cube is just six times that through one face:

$$\Phi_E = 6\left(\frac{Q}{6\varepsilon_0}\right) = \frac{Q}{\varepsilon_0}$$

The result shows that the electric flux Φ_E passing through a closed surface is proportional to the charge enclosed. In addition, the result further reinforces the notion that Φ_E is independent of the shape of the closed surface.

4.8.3 Gauss's Law for Gravity

What is the gravitational field inside a spherical shell of radius a and mass m?

Solution:

Since the gravitational force is also an inverse square law, there is an equivalent Gauss's law for gravitation:

$$\Phi_g = -4\pi G m_{\text{enc}} \tag{4.8.2}$$

The only changes are that we calculate gravitational flux, the constant $1/\varepsilon_0 \to -4\pi G$, and $q_{\text{enc}} \to m_{\text{enc}}$. For $r \le a$, the mass enclosed in a Gaussian surface is zero because the mass is all on the shell. Therefore the gravitational flux on the Gaussian surface is zero. This means that the gravitational field inside the shell is zero!

4.8.4 Electric Potential of a Uniformly Charged Sphere

An insulated solid sphere of radius a has a uniform charge density ρ. Compute the electric potential everywhere.

Solution:

Using Gauss's law, we showed in Example 4.4 that the electric field due to the charge distribution is

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}, & r > a \\ \dfrac{Qr}{4\pi\varepsilon_0 a^3}\hat{\mathbf{r}}, & r < a \end{cases} \tag{4.8.3}$$

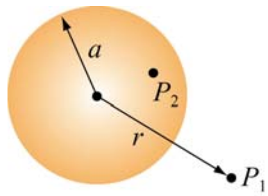


Figure 4.8.6

The electric potential at P_1 (indicated in Figure 4.8.6) outside the sphere is

$$V_1(r) - V(\infty) = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r'^2}\,dr' = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = k_e\frac{Q}{r} \tag{4.8.4}$$

On the other hand, the electric potential at P_2 inside the sphere is given by

$$\begin{aligned} V_2(r) - V(\infty) &= -\int_\infty^a drE(r > a) - \int_a^r E(r < a) = -\int_\infty^a dr\frac{Q}{4\pi\varepsilon_0 r^2} - \int_a^r dr'\frac{Qr}{4\pi\varepsilon_0 a^3}r' \\ &= \frac{1}{4\pi\varepsilon_0}\frac{Q}{a} - \frac{1}{4\pi\varepsilon_0}\frac{Q}{a^3}\frac{1}{2}\left(r^2 - a^2\right) = \frac{1}{8\pi\varepsilon_0}\frac{Q}{a}\left(3 - \frac{r^2}{a^2}\right) \\ &= k_e\frac{Q}{2a}\left(3 - \frac{r^2}{a^2}\right) \end{aligned} \tag{4.8.5}$$

A plot of electric potential as a function of r is given in Figure 4.8.7:

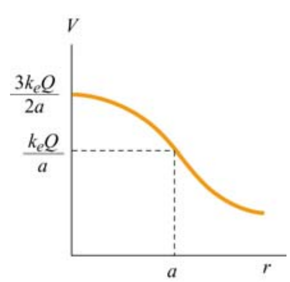


Figure 4.8.7 Electric potential due to a uniformly charged sphere as a function of r.

4.9 Conceptual Questions

1. If the electric field in some region of space is zero, does it imply that there is no electric charge in that region?

2. Consider the electric field due to a non-conducting infinite plane having a uniform charge density. Why is the electric field independent of the distance from the plane? Explain in terms of the spacing of the electric field lines.

3. If we place a point charge inside a hollow sealed conducting pipe, describe the electric field outside the pipe.

4. Consider two isolated spherical conductors each having net charge $Q > 0$. The spheres have radii *a* and *b*, where *b*>*a*. Which sphere has the higher potential?

4.10 Additional Problems

4.10.1 Non-Conducting Solid Sphere with a Cavity

A sphere of radius 2*R* is made of a non-conducting material that has a uniform volume charge density ρ. (Assume that the material does not affect the electric field.) A spherical cavity of radius *R* is then carved out from the sphere, as shown in the figure below. Compute the electric field within the cavity.

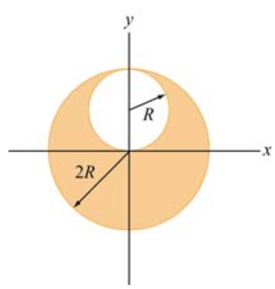


Figure 4.10.1 Non-conducting solid sphere with a cavity

4.10.2 P-N Junction

When two slabs of N-type and P-type semiconductors are put in contact, the relative affinities of the materials cause electrons to migrate out of the N-type material across the junction to the P-type material. This leaves behind a volume in the N-type material that is positively charged and creates a negatively charged volume in the P-type material.

Let us model this as two infinite slabs of charge, both of thickness *a* with the junction lying on the plane $z = 0$. The N-type material lies in the range $0 < z < a$ and has uniform charge density $+\rho_0$. The adjacent P-type material lies in the range $-a < z < 0$ and has uniform charge density $-\rho_0$. Thus:

$$\rho(x,y,z) = \rho(z) = \begin{cases} +\rho_0 & 0 < z < a \\ -\rho_0 & -a < z < 0 \\ 0 & |z| > a \end{cases}$$

(a) Find the electric field everywhere.

(b) Find the potential difference between the points P_1 and P_2. The point P_1 is located on a plane parallel to the slab a distance $z_1 > a$ from the center of the slab. The point P_2 is located on plane parallel to the slab a distance $z_2 < -a$ from the center of the slab.

4.10.3 Sphere with Non-Uniform Charge Distribution

A sphere made of insulating material of radius R has a charge density $\rho = ar$ where a is a constant. Let r be the distance from the center of the sphere.

(a) Find the electric field everywhere, both inside and outside the sphere.

(b) Find the electric potential everywhere, both inside and outside the sphere. Be sure to indicate where you have chosen your zero potential.

(c) How much energy does it take to assemble this configuration of charge?

(d) What is the electric potential difference between the center of the cylinder and a distance r inside the cylinder? Be sure to indicate where you have chosen your zero potential.

4.10.4 Thin Slab

Let some charge be uniformly distributed throughout the volume of a large planar slab of plastic of thickness d. The charge density is ρ. The mid-plane of the slab is the *y-z* plane.

(a) What is the electric field at a distance x from the mid-plane when $|x| < d/2$?

(b) What is the electric field at a distance x from the mid-plane when $|x| > d/2$?
[Hint: put part of your Gaussian surface where the electric field is zero.]

4.10.5 Electric Potential Energy of a Solid Sphere

Calculate the electric potential energy of a solid sphere of radius R filled with charge of uniform density ρ. Express your answer in terms of Q, the total charge on the sphere.

4.10.6 Calculating Electric Field from Electrical Potential

Figure 4.10.2 shows the variation of an electric potential V with distance z. The potential V does not depend on x or y. The potential V in the region $-1\,\text{m} < z < 1\,\text{m}$ is given in Volts by the expression $V(z) = 15 - 5z^2$. Outside of this region, the electric potential varies linearly with z, as indicated in the graph.

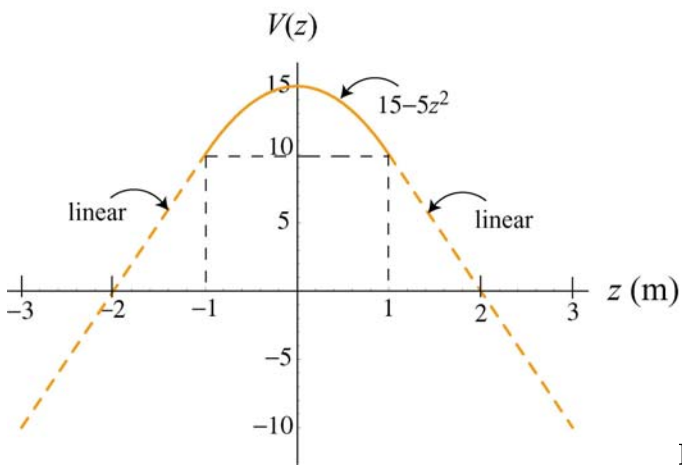


Figure 4.10.2

(a) Find an equation for the z-component of the electric field, E_z, in the region $-1\,\text{m} < z < 1\,\text{m}$.

(b) What is E_z in the region $z > 1$ m? Be careful to indicate the sign of E_z.

(c) What is E_z in the region $z < -1$ m? Be careful to indicate the sign of E_z.

(d) This potential is due a slab of charge with constant charge per unit volume ρ_0. Where is this slab of charge located (give the z-coordinates that bound the slab)? What is the charge density ρ_0 of the slab in C/m^3? Be sure to give clearly both the sign and magnitude of ρ_0.