8.022 (E&M) – Lecture 13

Topics:
- B’s role in Maxwell’s equations
- Vector potential
- Biot-Savart law and its applications

What we learned about magnetism so far...

- Magnetic Field B
  - Experiments: currents in wires generate forces on charges in motion
  - Force exerted on charge q with velocity v: \( \vec{F} = q \frac{v}{c} \times \vec{B} \)
  
  - Explanation: there must exist a magnetic field B
  - Special Relativity: B is just E seen from another reference frame...

- Ampere’s Law: \( \oint \vec{B} \cdot d\vec{s} = \frac{4\pi}{c} I_{encl} \)
  
  - Application: B generated by current in a wire: \( \ddot{\vec{B}} = \frac{2I}{cr^2} \phi \)
Divergence of B

- Consider the B produced by a wire of current: \( \vec{B} = \frac{2I}{cr} \hat{\phi} \)
- Calculate its divergence in Cartesian coordinates:
  
  Given \( r = \sqrt{x^2 + y^2} \) and \( \hat{\phi} = \hat{y} \cos \phi - \hat{x} \sin \phi = \frac{x\hat{y}}{\sqrt{x^2 + y^2}} - \frac{y\hat{x}}{\sqrt{x^2 + y^2}} \) \( \Rightarrow \)
  
  \[ \vec{B} = \frac{2I}{cr} \left( \frac{x\hat{y}}{x^2 + y^2} \cdot \frac{y\hat{x}}{x^2 + y^2} \right) \Rightarrow \nabla \cdot \vec{B} = \frac{2I}{cr} \left( \frac{2yx}{(x^2 + y^2)^2} - \frac{2xy}{(x^2 + y^2)^2} \right) = 0 \]

- This is a general property of the magnetic field: \( \nabla \cdot \vec{B} = 0 \)
- Similar equation for E: \( \nabla \cdot \vec{E} = 4\pi \rho \)
  - The divergence of E is related to the density of electric charges
  - The divergence of B must be related to the density of magnetic charges
  \( \rightarrow \) Magnetic monopole don’t exist
(There may be magnetic monopoles leftover from the Early Universe, but never observed experimentally so far)

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Ampere’s law in differential form

- Apply Stoke’s theorem to Ampere’s law:
  
  \[ \oint_C \vec{B} \cdot d\vec{S} = \frac{4\pi}{c} I_{encl} \]
  
  \[ \oint_C \vec{B} \cdot d\vec{S} = \int_S \nabla \times \vec{B} \cdot d\vec{S} = \frac{4\pi}{c} \int_S \vec{J} \cdot d\vec{S} \]
  
  \[ \int_S \left( \nabla \times \vec{B} - \frac{4\pi}{c} \vec{J} \right) \cdot d\vec{S} = 0 \text{ for any surface} \]
  
  \( \rightarrow \) Ampere’s law in differential form: \( \nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} \)

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Toward Maxwell’s equations

- Let’s collect all the equations in differential form that we found so far:
  \[
  \begin{align*}
  \nabla \cdot \vec{E} &= 4\pi \rho & \text{Relates } E \text{ and charge density } (\rho) & \text{- Gauss} \\
  \nabla \cdot \vec{B} &= 0 & \text{No magnetic monopoles!} \\
  \nabla \times \vec{E} &= 0 & \text{E is a conservative field} \\
  \nabla \times \vec{B} &= \frac{4\pi}{c} \vec{J} & \text{Relates } B \text{ and its sources (J)} & \text{- Ampere}
  \end{align*}
  \]

- Not complete Maxwell’s equations yet, but we are getting closer...

Vector potential \(A\)

- Definition of potential for electric field:
  - \(\phi(P) = \text{work needed to move a unit charge from reference to } P\)
  - Relationship between \(\phi\) and \(E\): \(\vec{E} = -\nabla \phi\)
  - Hidden advantage:
    - If \(\vec{E} = -\nabla \phi \Rightarrow \nabla \times \vec{E} = 0\) because \(\nabla \times (\nabla \phi) = 0 \ \forall \phi\)

- Can we introduce something similar for \(B\)?
  - Goal: enforce \(\text{div } B = 0\)
  - Since \(\nabla \cdot \vec{\nabla \times \vec{f}} = 0\) for any \(\vec{f}\), we define
    \[
    \vec{B} \equiv \nabla \times \vec{A}
    \]

- \(A\) is called “vector potential” in analogy with \(\phi\)
- \(A\) is not connected to work or energy (but to angular momentum)
Non Uniqueness

- **Electrostatics:** given a charge distribution and boundary conditions $\rightarrow$ potential $\phi$ is uniquely identified
- **Magnetism:** does it work the same for $A$? No, there are infinite number of $A$ corresponding to a single $B$
  - Example: $\vec{B} = B_y \hat{z}$. Find $\vec{A}$ that creates this $\vec{B}$ field.

Requirements:

$$
\begin{align*}
B_y &= \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z} = 0 \\
B_z &= \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial x} = 0 \\
B_x &= \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial y} = B_0
\end{align*}
$$

Q: what current creates this $B$?

- We are given one “coupon” to simplify equations when needed

Poisson’s equation for $A$

- **Electrostatics:**

$$
\begin{align*}
\vec{E} &= -\vec{\nabla} \phi \\
\vec{\nabla} \cdot \vec{E} &= 4\pi \rho
\end{align*}
$$

$\Rightarrow \nabla^2 \phi = -4\pi \rho$ Poisson’s equation

- **Magnetism:**

$$
\begin{align*}
\vec{B} &= \vec{\nabla} \times \vec{A} \\
\vec{\nabla} \times \vec{B} &= \frac{4\pi}{c} j
\end{align*}
$$

$\Rightarrow \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \frac{4\pi}{c} j$ $\Rightarrow \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \frac{4\pi}{c} j$

We used the identity: $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$ (Pset#7)

- Use your coupon now!

Choosing $\vec{\nabla} \cdot \vec{A} = 0$ $\Rightarrow \nabla^2 \vec{A} = -\frac{4\pi}{c} j$
Solving Poisson’s equation for $A$

How do you solve $\nabla^2 \vec{A} = -\frac{4\pi}{c} \vec{J}$?

Think of it in cartesian coordinates:

\[
\begin{align*}
\nabla^2 A_x & = -\frac{4\pi}{c} J_x \\
\nabla^2 A_y & = -\frac{4\pi}{c} J_y \\
\nabla^2 A_z & = -\frac{4\pi}{c} J_z 
\end{align*}
\]

Remember Poisson’s equation $\nabla^2 \phi = -4\pi \rho$ and its solution $\phi = \frac{\int \rho}{r}$

Same as our new equation if replace $\phi \rightarrow \vec{A}$ and $\rho \rightarrow \frac{\vec{J}}{c}$

For current flowing in a wire: $\vec{A} = \frac{l}{c} \int_{\text{wire}} \frac{d\vec{l}}{r}$

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Biot-Savart Law

Find $\vec{B}$ produced from current knowing that $\vec{A} = \frac{l}{c} \int_{\text{wire}} \frac{d\vec{l}}{r}$

$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \frac{l}{c} \int_{\text{wire}} \frac{d\vec{l}}{r} = \frac{l}{c} \int_{\text{wire}} \vec{\nabla} \times \frac{d\vec{l}}{r}$

Using the fact that $\nabla \times (a \vec{b}) = a(\nabla \times \vec{b}) + (\nabla a) \times \vec{b}$:

$= \frac{l}{c} \left[ \int_{\text{wire}} \frac{1}{r} (\vec{\nabla} \times d\vec{l}) + \vec{\nabla} \frac{1}{r} \times d\vec{l} \right] = \frac{l}{c} \int_{\text{wire}} \frac{1}{r} (\vec{\nabla} \times d\vec{l}) + \vec{\nabla} \frac{1}{r} \times d\vec{l}$

Since $\vec{\nabla} \times d\vec{l} = 0$ and $\vec{\nabla} \frac{1}{r} = -\frac{\vec{r}}{r^2}$:

$\Rightarrow \vec{B} = \frac{l}{c} \int_{\text{wire}} \frac{d\vec{l}}{r} \times \frac{\vec{r}}{r^2}$
Biot-Savart Law: illustration

- Biot-Savart: \( \mathbf{dB} = \frac{l}{c} \mathbf{dI} \times \frac{\mathbf{r}}{r^2} \)

- \( \mathbf{dB} \) is perpendicular to current and to radial direction
- E.g.: if you have \( \mathbf{dl} // x, \mathbf{r} // y \rightarrow \mathbf{B} // z \)

Application of Biot-Savart: B from loop of current

- Calculate \( \mathbf{B} \) created by a loop of current
  - Radius: \( R \)
  - Distance from center of the loop: \( z \)
- Solution on axis
  - Apply Biot-Savart
  - Determine direction of \( \mathbf{dB} \)
  - Symmetry \( \rightarrow \) only component \( // z \) survives

\[
\mathbf{B} = \int_{\text{wire}} (\mathbf{d\tilde{B}})_z = \int_{\text{wire}} \frac{l}{cR^2} \left| \mathbf{d\tilde{l}} \times \mathbf{\hat{r}} \right| \sin \theta \\
\left| \mathbf{d\tilde{l}} \times \mathbf{\hat{r}} \right| = \left| \mathbf{d\tilde{l}} \right| = R d\phi; \quad \sin \theta = R / r; \quad r = \sqrt{R^2 + h^2}
\]

\[
\mathbf{\tilde{B}} = \frac{l}{cR^2} R \sin \theta \int_0^{2\pi} d\phi \mathbf{\hat{z}} = \frac{2\pi l R^2}{c(R^2 + z^2)^{3/2}} \mathbf{\hat{z}} \quad \Rightarrow \quad \mathbf{\tilde{B}}_{\text{loop center}} = \frac{2\pi l R^2}{cR} \mathbf{\hat{z}}
\]
Application of Biot-Savart: B from solenoid

- What if we stack a N rings over a length L?
- Use result of single loop + superposition:

Single ring: \( \text{d}B = \frac{2\pi R^2}{c(R^2 + z^2)^{3/2}} \text{d}l \)

Integrate on all rings (in the middle of the solenoid)

\[
\vec{B} = \int_{L/2}^{L/2} \frac{2\pi R^2}{c(R^2 + z^2)^{3/2}} n/\text{d}z = \frac{2\pi n/}{c} \int_{L/2}^{L/2} \frac{R^2 \text{d}z}{(R^2 + z^2)^{3/2}}
\]

\[
= \frac{2\pi n/}{c} \frac{2L}{\sqrt{L^2 + 4R^2}}
\]

With \( n=N/L \)

- For \( L\gg R \):

\[
\vec{B} = \frac{4\pi n/}{c}
\]

Solenoid and Ampere’s law

- One can prove that \( B \) outside the solenoid is =0
- Ampere can be used to simply prove that \( B \) does not depend on \( r \):

\[
\oint \vec{B} \cdot \text{d}\vec{l} = \frac{4\pi}{c} I_{\text{enc}}
\]

Since \( \vec{B} \) is \( I/z \) and present only inside the solenoid:

\[
B(r)L = \frac{4\pi}{c} NI \Rightarrow B(r) = \frac{4\pi N I}{c L} = \frac{4\pi}{c} nI \text{ no dependence on } R
\]
Solenoid’s magnetic field: demos

- Expected:

- Can we test this experimentally?
  - G12: B from a single wire using iron filings
  - G13: B from 2 wires
  - G16: B inside solenoid

More demos on magnetic fields

- More demos:
  - G14: map B around a wire using a compass
  - G9a: collapsing solenoid
    - Can you explain what’s happening?
  - G18: Long solenoid
    - Long solenoid with \( N_{\text{turn}} = 2760 \), \( I = 4.5 \text{ mA} \), length = 46 cm
      - \( R = 10 \Omega \), \( L = 128 \text{ mH} \)
    - What is B?

\[
B = \frac{4\pi}{c} n I = \frac{4\pi}{3 \cdot 10^{10}} \frac{2760}{50} 4.5 = 230 \cdot 10^{-8} \text{ Gauss} ???
\]

- Verify with Hall probe
Thompson’s experiment: variation

- Variation on a theme: instead of canceling effects of E and B, one could tune the fields and measure the radius of curvature of the electron beam.

- Parameters of the problem:
  - $V = 300 \text{ V}$
  - $I = 1.4 \text{ A}$
  - $R = 5 \text{ cm}$

- Solution:
  - $e/m = 2.02 \times 10^{11} \text{ C/Kg}$  (cfr: $1.76 \times 10^{11} \text{ C/Kg}$)

Summary and outlook

- Today:
  - Toward Maxwell's equations: $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$

  - Vector Potential: $\vec{B} \equiv \vec{\nabla} \times \vec{A}$

  - Biot-Savart Law: $d\vec{B} = \frac{I}{c} d\vec{l} \times \frac{\hat{r}}{r^2}$

- Next time:
  - What happens when B varies in time?
    - Faraday’s and Lenz’s laws and their applications