8.022 (E&M) – Lecture 19

Topics:
- The missing term in Maxwell’s equation
  - Displacement current: what it is, why it’s useful
- The complete Maxwell’s equations
  - And their solution in vacuum: EM waves

Maxwell’s equations so far

\begin{align*}
\n \nabla \cdot \vec{E} &= 4\pi \rho \\
\nabla \cdot \vec{B} &= 0 \\
\n\nabla \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\
\n\nabla \times \vec{B} &= \frac{4\pi}{c} \vec{J}
\end{align*}

\leftrightarrow \text{ Gauss’s law: relates } E \text{ and charge density } (\rho) \\
\leftrightarrow \text{ Magnetic field lines are always closed!} \\
\leftrightarrow \text{ Faraday’s law: change in } B \text{ flux creates e.m.f. } (E) \\
\leftrightarrow \text{ Ampere’s law: relates } B \text{ and its sources } (J)

Is this set of equations completely consistent? 
Not quite...
Maxwell’s equations so far (2)

\[ \begin{align*}
\nabla \cdot \vec{E} &= 4\pi \rho \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \frac{4\pi}{c} \vec{j}
\end{align*} \]

- Is this set of equations consistent? Not quite…
  - Take the divergence of Ampere's law
    - \[ \nabla \cdot \left( \frac{4\pi}{c} \vec{j} \right) = \frac{4\pi}{c} \nabla \cdot \vec{j} = \frac{4\pi}{c} \frac{\partial \rho}{\partial t} \text{ (using continuity equation)} \]
    - \[ \nabla \cdot \nabla \times \vec{B} = 0 \quad (\nabla \cdot \nabla \times \vec{v} \text{ is ALWAYS 0!}) \]

Ampere's law works only when \( \frac{\partial \rho}{\partial t} = 0 \) which works in most cases but not always: Ampere's law is incomplete!

Fixing the inconsistency

- Since \( \nabla \cdot \nabla \times \vec{v} \equiv 0 \) we need to add some term to the right hand side to that its divergence will be identically 0
- Generalized Ampere’s law: \( \nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} + \vec{F} \)
- What is \( \vec{F} \)? We know that its divergence must be =0:
  \[ \nabla \cdot \left( \frac{4\pi}{c} \vec{j} + \vec{F} \right) = 0 \Rightarrow \nabla \cdot (c\vec{F}) = 4\pi \frac{\partial \rho}{\partial t} \]
  \[ \Rightarrow \nabla \cdot (c\vec{F}) = 4\pi \frac{\partial \rho}{\partial t} \quad \text{Similar to Gauss's law!} \]
- Take time derivative of Gauss’s law:
  \[ \frac{\partial}{\partial t} (\nabla \cdot \vec{E}) = 4\pi \frac{\partial \rho}{\partial t} \Rightarrow \frac{\partial}{\partial t} (\nabla \cdot \vec{E}) = \nabla \cdot \left( \frac{\partial \vec{E}}{\partial t} \right) = \nabla \cdot (c\vec{F}) \text{ time and space derivatives commute} \]
  \[ \Rightarrow \vec{F} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \]
Displacement currents

- Generalized Ampere's equation
  \[ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \]

- This can also be written as:
  \[ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} (\vec{J} + \vec{J}_d) \]

- With \( J_d = \) displacement current (density):
  \[ J_d = \frac{1}{4\pi} \frac{\partial \vec{E}}{\partial t} \]

- What is the \( J_d \)?
  - Not a real current: does not describe charges flowing through some region
  - But it acts like a real current: whenever we have changing E field, we can treat its effect as if due to a real current \( J_d \)

What is a displacement current?

- Consider a current flowing in a circuit and charging a capacitor \( C \)

  ![Diagram of current flowing and charging a capacitor](Image)

- Standard integral Ampere's law:
  \[ \oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} \int J_{\text{encl}} = \frac{4\pi}{c} \int_S \vec{j} \cdot d\vec{a} \]

  - Let's choose the path \( C \) and the surface \( S \) as in the drawing above:
    - It all makes sense!
  - Now choose the same path \( C \) but the surface \( S' \) (ok by Stokes...)
    - No standard current \( \vec{J} \) through the surface (no charge crosses \( C! \))
    - But there is a flux of displacement current \( J_d \) through the plates!
What is a displacement current? (2)

- We can use the generalized Ampere’s Law:
  \[ \oint \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi}{c} (I_{\text{enc}} + I_d) \]

  with \[ I_d = \int_S \int_{S'} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{4\pi} \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} = \frac{1}{4\pi} \int_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{4\pi} \oint \frac{\partial \Phi_E}{\partial t} \]

- The displacement current is related to the change over time of the flux of the electric field.
- In the example above, the electric field \( E \) is the one produced in between the plates of the capacitor \( C \).

What is a displacement current? (3)

- The electric field \( E \):
  - Points in the same direction of the current (+x)
  - At a given instant in time: \( E = \frac{4\pi Q}{A} \hat{x} \)

- The flux of \( E \) will then be: \( \Phi_E = 4\pi Q \) (yes, Gauss’s law!)

- The rate of the change if this flux is: \( \frac{\partial \Phi_E}{\partial t} = 4\pi \frac{\partial Q}{\partial t} = 4\pi I \)
  - Where \( I \) is the current that is charging the capacitor

- Comparing this with results in the previous page:
  \[ I_d = \int_S \oint_{S'} \mathbf{j} \cdot d\mathbf{a} = \int_S \oint_{S'} \mathbf{j} \cdot d\mathbf{a} = I \]

  \[ \rightarrow \text{generalized Ampere’s Law is valid no matter what surface we use} \]
The importance of displacement currents

- When we examined the following circuit:

we said the same current I was flowing in each circuit element.
- How is it possible? No current flows through the plates of a capacitor!
  - Displacement currents fix this inconsistency!
  - Displacement current “continues” the “real” current across the capacitors ensuring the validity of Kirchoff’s laws.

Maxwell’s equations (complete!)

\begin{align*}
\nabla \cdot \vec{E} &= 4\pi \rho \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}
\end{align*}

\begin{align*}
\nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \mu_0 \vec{J} + \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{align*}

NB: when Maxwell introduced the term dE/dt in the generalized Ampere’s law, his arguments were based purely on symmetry
- Yes, he was a theorist!
Maxwell’s equations: integral form

\[
\begin{align*}
\Phi_E &= \int_S \vec{E} \cdot d\vec{a} = 4\pi Q_{enc} \quad \text{(Gauss's law)} \\
\Phi_B &= 0 \\ \text{emf} &= \oint_C \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{\partial \Phi_B}{\partial t} \quad \text{(Faraday's law)} \\
\oint_C \vec{B} \cdot d\vec{l} &= \frac{4\pi}{c} (\vec{I} + \vec{I}_a) \quad \text{(Generalized Ampere's law)}
\end{align*}
\]

where the currents \( \vec{I} \) and \( \vec{I}_a \) are defined as \( \vec{I} = \int_S \vec{J} \cdot d\vec{a} \) and

\[
\vec{I}_a = \frac{1}{4\pi} \frac{\partial \Phi_E(S)}{\partial t}
\]

3 good reasons to remember Maxwell’s equations

1) They compactly and beautifully summarize all the E&M we learned so far!

2) You will see them on T shirts for the rest of your life at MIT: better to get familiar with them ASAP!

3) On the first day of 8.03 next semester you will be asked to write them down on a piece of paper to check what you learned in your first semester at MIT: save your honor (and mine)
Displacement current: application

Consider the following RC circuit:
- As C charges up, \( I_d \) flows
- \( I_d \) induces \( B \) inside the plates
- Assuming cylindrical plates of radius \( a \)

Calculate \( B \) inside the plates

1) Find \( \vec{E}(t) \):

\[
\vec{E}(t) = 4\pi \sigma \frac{Q(t)}{\pi a^2}
\]

2) Displ. current density:

\[
\vec{j}_d = \frac{1}{4\pi} \frac{\partial \vec{E}}{\partial t} = \frac{1}{4\pi} \frac{\partial E(t)}{\partial t} = \frac{1}{\pi a^2} \frac{\partial Q(t)}{\partial t} = \frac{I(t)}{\pi a^2}
\]

3) Remember that \( I(t) = \frac{V}{R} e^{-t/RC} \)

4) Magnetic field inside the plate (Ampere's law):

\[
\oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} \int \vec{j}_d \cdot d\vec{a}
\]

\[
\Rightarrow \vec{B}(r) = \frac{2\pi V}{ca^2 R} e^{-t/RC}
\]

Maxwell equations in vacuum

What happens when we write Maxwell's equations in vacuum?
- Vacuum: no sources, \( \rho = 0 \) and \( J = 0 \)

\[
\begin{align*}
\nabla \cdot \vec{E} &= 4\pi \rho \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}
\end{align*}
\]

\[
\begin{align*}
\nabla \cdot \vec{E} &= 0 \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t}
\end{align*}
\]

- Except for a – sign, these equations are exquisitely symmetric!
- Consequence: an electric field \( E \) varying in time will create a magnetic filed \( B \); a \( B \) varying in time creates a \( E \); \( E \) and \( B \) are intimately related!
Maxwell equations in vacuum: solution

- How to solve these equations?
  - Uncouple them!
  - Separate E and B in equations
- How?
  - Take the curl of equations (3) and (4)
  - Use other equations as needed

\[ \begin{align*}
\vec{\nabla} \cdot \vec{E} &= 0 \quad (1) \\
\vec{\nabla} \cdot \vec{B} &= 0 \quad (2) \\
\vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (3) \\
\vec{\nabla} \times \vec{B} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad (4)
\end{align*} \]

Start from (3):

Left: \( \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\vec{\nabla}^2 \vec{E} \) (since \( \vec{\nabla} \cdot \vec{E} = 0 \) in vacuum)

Right: \( \vec{\nabla} \times \left( -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \right) = -\frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \times \vec{B} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \) (using (4))

\[ \Rightarrow \vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \]

Maxwell equations in vacuum: solution

- Now repeat the procedure starting from \( \vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \) (4)

Left: \( \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{B}) - \vec{\nabla}^2 \vec{B} = -\vec{\nabla}^2 \vec{B} \) (since \( \vec{\nabla} \cdot \vec{B} = 0 \) in vacuum)

Right: \( \vec{\nabla} \times \left( \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right) = \frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \times \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} \) (using (3))

\[ \Rightarrow \vec{\nabla}^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} \]

- This is a special case of a known equation: the wave equation:
  \[ \vec{\nabla}^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} \]
  where \( f = f(x \pm vt) \)

NB: we are restricting ourselves to the 1D case; extension to 3D next lecture
Solution of wave equation: prove

- Prove that \( f = f(x \pm vt) \) is a solution of the wave equation
- Just calculate time and space derivatives.
  - Keep in mind that \( \ddot{\psi}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \)
- Define \( u = x \pm vt \)

\[
\frac{\partial f(x \pm vt)}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} = \pm v \frac{\partial f}{\partial u} \Rightarrow \frac{\partial^2 f(x \pm vt)}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial u^2} \\
\frac{\partial f(x \pm vt)}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} \Rightarrow \frac{\partial^2 f(x \pm vt)}{\partial t^2} = \frac{\partial^2 f}{\partial u^2}
\]

Plug the above results into the equation \( \Rightarrow \frac{\partial^2 f}{\partial u^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial u^2} \Rightarrow \) identity!

- As we wanted to prove!

Wave equation solution

- What is a function such as \( f = f(x - vt) \)?
  - Assume \( v = 1 \) cm/s

- At time \( t=0 \):
  - Position of the max: \( x_0 \)

- At time \( t=1 \) s:
  - The peak still occurs when the argument of \( f \) is \( x_0 \)
  - But since the time is not 0
    \( \Rightarrow \) the function will be shifted in \( x \) by "vt"=1 cm
  - Position of the max: \( x_1 = x_0 + 1 \)

\( f = f(x - vt) \) represents a wave traveling in the +x direction with velocity \( v \)
Wave equation solution

- What is a function such as $f = f(x + vt)$?
  - Assume $v=1$ cm/s

- At time $t=0$:
  - Position of the max: $x_0$

- At time $t=1$ s:
  - The peak still occurs when the argument of $f$ is $x_0$
  - But since the time is not 0
  - The function will be shifted in $x$ by “vt”=1 cm
  - Position of the max: $x_1=x_0-1$

$f = f(x + vt)$ represents a wave traveling in the -x direction with velocity $v$

EM waves

- Wave equation: $\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$
- Solution: $f = f(x \pm vt)$
  - Any function of argument $x \pm vt$
  - These solution represent waves traveling with velocity $v$
    - $x - vt$ represents a wave traveling in the +x direction
    - $x + vt$ represents a wave traveling in the -x direction

- Maxwell’s equation: $\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$
  - Same equation! Only difference: $v=c$
  - Solution: EM waves traveling with speed of light
    - The light IS an EM wave!!!
EM waves in SI

- This same result looks much more interesting in SI.
- Maxwell's equations in SI:
  \[ \nabla \cdot f = \mu_0 \epsilon_0 \frac{\partial^2 f}{c^2} \]
  where \( \epsilon_0 \) is the permittivity of free space
  and \( \mu_0 \) is the permeability of free space
- Maxwell's equations tell us what the velocity of an EM wave is:
  \[ v = 1 / \sqrt{\mu_0 \epsilon_0} \]
- \( \epsilon_0 \) and \( \mu_0 \) can be measured → we can predict velocity of EM waves:
  \( \epsilon_0 = 8.85418 \times 10^{-12} \text{Coulomb}^2 \text{Newton}^{-1} \text{meter}^{-2} \)
  \( \mu_0 = 4 \pi \times 10^{-7} \text{Newton} \text{sec}^2 \text{Coulomb}^{-2} \)
  \[ v = 2.998 \times 10^8 \text{m/s} \] which is the speed of light!
- Maxwell was the first to realize that E&M equations were leading to a wave equation that was propagating at the speed of light: light is an EM wave!

How to measure c (demo A4)

- Experimental setup: a neon laser beam is sent into a beam splitter. Part of it is reflected and part of it is refracted first and then reflected by a mirror.
- Difference in path between the 2 beams:
  \[ \approx 17.15 \text{m} \times 2 = 34.3 \text{meters} \]
- Measure the delay of channel 2 wrt channel 1 on the scope: 116 ns
  \[ v = \frac{34.3 \text{m}}{116 \text{ns}} = 2.96 \times 10^8 \text{m/s} \]
Summary and outlook

- **Today:**
  - Complete Maxwell’s equations
    - The missing term leads to displacement currents
  - Solution of Maxwell’s equations in vacuum
    - Wave equation $\rightarrow$ light is an EM radiation

- **Next time:**
  - Properties of EM radiation
  - Polarization and scattering of light