Last time

- Solution of Maxwell’s equations in vacuum
  \[ \nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \]
- Solution of wave equation \( f(r \pm ck_t) \) can be expressed as linear combination of plane waves:
- Properties of plane waves: \( \vec{E} = \vec{E}_0 \sin(\vec{k} \cdot \vec{r} - \omega t); \vec{B} = \vec{B}_0 \sin(\vec{k} \cdot \vec{r} - \omega t) \)
  - They travel at the speed of light // to k (wave vector)
  - E, B and k are always perpendicular to each other
  - Amplitude of E and B are the same in cgs
- Polarization of EM waves
  - Linear: when the direction of \( \vec{E}_0 \) is constant in time
  - Circular: when the vector \( \vec{E}_0 \) describes a circle over time
  - Elliptical: all the situations in between these 2 cases
- Today we will complete the study of these properties...
EM Energy

- EM radiation carries energy
  - Obvious if you think about the fact that is the light from the sun that keeps us warm...
- How does this energy propagate?
  - Consider a volume \( V \) of surface \( A \) containing \( E \) and \( B \)

\[
E = \frac{\text{energy}}{\text{volume}} = \frac{1}{8\pi} (\vec{E} \cdot \vec{B} + \vec{B} \cdot \vec{B})
\]

Total energy: \( \int \text{udV} = \frac{1}{8\pi} \int (\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}) \text{dV} \)

The Poynting vector

- How does total derivative change over time?

\[
\frac{\partial U}{\partial t} = \frac{1}{8\pi} \frac{\partial}{\partial t} \int (\vec{E} \cdot \vec{B} + \vec{B} \cdot \vec{B}) \text{dV} = \frac{1}{4\pi} \int (\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}) \text{dV}
\]

Remembering that in vacuum: \( \vec{V} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \) and \( \vec{V} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \)

\[
\Rightarrow \frac{\partial U}{\partial t} = \frac{c}{4\pi} \int (\vec{V} \times \vec{B} \cdot \vec{E} - \vec{V} \times \vec{E} \cdot \vec{B}) \text{dV}
\]

Remembering that \( \vec{V} \cdot (\vec{E} \times \vec{B}) = -\vec{E} \cdot (\vec{V} \times \vec{B}) + \vec{B} \cdot (\vec{V} \times \vec{E}) \)

\[
\Rightarrow \frac{\partial U}{\partial t} = -\frac{c}{4\pi} \int \vec{V} \cdot (\vec{B} \times \vec{E}) \text{dV} = -\int \vec{V} \cdot \vec{S} \text{ dV}
\]

where we defined the Poynting vector as \( \vec{S} = \frac{c}{4\pi} \vec{B} \times \vec{E} \)
Interpretation of Poynting vector

- **Given:**
  \[ \frac{\partial U}{\partial t} = -\int \nabla \cdot \vec{S} \, dV \quad \Rightarrow \quad \frac{\partial U}{\partial t} = -\int_{A} \vec{S} \cdot d\vec{a} = -\Phi_{S}(A) \]

  - The rate of change of EM energy in the volume V is given by the flux of the Poynting vector S through the surface A
    - Minus sign: dA points outward \( \Rightarrow \) U increases when S is opposite to dA

- **Interpretation of Poynting vector:**
  - \( \vec{S} = \frac{c}{4\pi} \vec{B} \times \vec{E} \) points in the direction of the EM energy flow
    - Remember that \( \vec{E}_a \times \vec{B}_a = |\vec{E}_a| \hat{k} \)
  - The flux of S through a surface gives the power through A
    - Power through A: \( \int_{A} \vec{S} \cdot d\vec{a} \)

---

Poynting vector: dimensional analysis

- **What are the units of the Poynting vector?**
  \[
  \left[ \vec{S} \right] = \left[ \frac{c}{4\pi} \vec{E} \times \vec{B} \right] = \left[ c \vec{E} \vec{B} \right] = \left[ c \vec{E} \right] = \left[ \text{Length} \times \text{Energy} \right]
  \]
  \[
  \left[ c \right] = \frac{\text{Length}}{\text{Time}}
  \]
  \[
  \text{From } u = \frac{1}{8\pi} (\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}) \Rightarrow \left[ \vec{E} \right] = \frac{\text{Energy}}{\text{Volume}}
  \]
  \[
  \Rightarrow \left[ \vec{S} \right] = \frac{\text{Length} \times \text{Energy}}{\text{Time} \times \text{Volume}} = \frac{\text{Energy}}{\text{Time} \times \text{Area}} = \frac{\text{Power}}{\text{Area}}
  \]

- Expected if the flux of S is the power through area A
- In cgs: \([S]=\text{erg s}^{-1} \text{ cm}^{-2}\)
- NB: Magnitude of S is known as Intensity I
  - Intense source of radiations emit a lot of power per unit area
Applications: plane waves

Consider a linearly polarized plane wave:
\[
\begin{aligned}
\vec{E} &= E_0 \cos(kz - \omega t) \hat{x} \\
\vec{B} &= B_0 \cos(kz - \omega t) \hat{y}
\end{aligned}
\]

Poynting vector associated with it:
\[
\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{c}{4\pi} E_0^2 \sin^2(kz - \omega t) \hat{k}
\]

This can be compared to the energy density of the wave:
\[
u = \frac{1}{8\pi} (\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}) = \frac{1}{4\pi} E_0^2 \sin^2(kz - \omega t)
\]

\[
\Rightarrow \vec{S} = \nu \hat{c} = \nu \hat{k}
\]

This is similar to \( \int = \rho \vec{v} \)
\( \Rightarrow \) another way to show that \( S \) tells us about the flow of energy!

Usually the oscillation is very fast (e.g.: visible\( \sim 10^{14} \) Hz) \( \Rightarrow \) all that matters is the average energy density \( \langle S \rangle \) and intensity \( \langle I \rangle \):
\[
\langle S \rangle = \frac{c k}{8\pi} E_0^2; \quad \langle I \rangle = \frac{c}{8\pi} E_0^2
\]

Application 2: Dipole radiation

Radiation emitted by a dipole oriented along the z axis in spherical coordinates:
\[
\begin{aligned}
\vec{E} &= \frac{\alpha^2 p}{c^2} \sin \theta \frac{\sin(kr - \omega t)}{r} \hat{\theta} \\
\vec{B} &= \frac{\alpha^2 p}{c^2} \sin \theta \frac{\sin(kr - \omega t)}{r} \hat{\phi}
\end{aligned}
\]

\( \text{NB: this solution is only valid for } r >> \lambda = 2\pi/k \)

This is the Radiation propagates radially, some angular dependence too.

Poynting vector:
\[
\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{1}{4\pi c^2} \frac{\omega^2 p^3 \sin^2 \theta \sin^2(kr - \omega t)}{r^2} \hat{r}
\]

\[
\Rightarrow \langle \vec{S} \rangle = \frac{\omega^2 p^3}{8\pi r^2 c^2} \sin^2 \theta \hat{r}
\]

\( \text{NB: Poynting vector (and I) falls as } 1/r^2: \text{ this should be intuitive. Why?} \)
Dipole radiation: cont.

- Draw a sphere of radius \( R \) around the dipole centered in origin:
  - NB: \( R \gg d \)
- Compute power radiated through the sphere:
  \[
  \left\langle \frac{\partial U}{\partial t} \right\rangle = \int_S \left\langle \mathbf{S} \cdot d\mathbf{a} \right\rangle = \int_S \frac{\omega^2 \rho^2}{8\pi R^2 c^2} \sin^2 \theta \hat{r} \cdot d\mathbf{a}
  \]
  
  Since \( d\mathbf{a} = R^2 \sin \theta d\phi \hat{r} : 
  \[
  \left\langle \frac{\partial U}{\partial t} \right\rangle = \frac{\omega^2 \rho^2}{8\pi R^2 c^2} \int_0^{2\pi} d\phi \int_0^{\pi/2} R^2 \sin^2 \theta d\theta
  \]

  Since \( \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{4}{3} \Rightarrow \left\langle \frac{\partial U}{\partial t} \right\rangle = \frac{\omega^2 \rho^2}{3c^2} \) (Larmor formula)

- NB: power through sphere does not depend on \( R \)
  - Why? \( S \) falls as \( 1/r^2 \), area increases as \( r^2 \)

  \( \rightarrow \) Power through \( S \) (flux through \( S \)) is constant: Energy is conserved

Application 3: capacitor

- The Poynting vector applies to ANY situation in which both \( E \) and \( B \) appear, not just when we have radiation
- Example: charging capacitor
  \[
  \mathbf{E} = -\frac{4\pi Q}{A} \hat{z} = -\frac{4Q}{a^2} \hat{z}
  \]

  From generalized Ampere law: \( \mathbf{B}(r) = \frac{2I r}{ca^2} \hat{\phi} \)

- Calculate Poynting vector:
  \[
  \mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \frac{c}{4\pi} \frac{4Q}{a^2} \frac{2I r}{ca^2} \hat{z} \times \hat{\phi} = \frac{2Qr}{\pi a^3} (-\hat{r})
  \]

- NB: what is important here is the direction of \( S \):
  - \( S \) points into the center of the capacitor as it should: the plates are charging up!
Momentum carried by EM wave

- Since EM carry energy it’s not surprising that they carry momentum as well.
- In relativity, E and p are related by \( E^2 = c^2 \left| \vec{p} \right|^2 + m^2 c^4 \).
- For EM radiation, m=0:
  \[ E^2 = c^2 \left| \vec{p} \right|^2 \Rightarrow \dot{p} = \frac{U}{c} \]
- Remember that
  \[ \dot{S} = \frac{\text{Power}}{\text{Area}} = \frac{\text{Energy}}{\text{Time Area}} = \frac{\text{Energy/c}}{\text{Time Area}} = \frac{\text{Momentum}}{\text{Time Area}} \]
- Dimensional analysis will also tell us that:
  \[ \frac{\dot{S}}{c} = \frac{\text{Momentum}}{\text{Time Area}} = \frac{\text{Force}}{\text{Area}} = \text{Pressure} \]

\( \Rightarrow \) Radiation exerts pressure

---

Summary on Poynting vector

- Energy flux: Energy / area / unit time
- Energy density \( u \): Energy / unit volume
- Momentum flux: Momentum / area / unit time
- Momentum density: Momentum / unit volume

<table>
<thead>
<tr>
<th>Flux</th>
<th>Energy</th>
<th>Momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{S} = \frac{c}{4\pi} \vec{B} \times \vec{E} )</td>
<td>( \frac{\dot{S}}{c} \text{(same as pressure)} )</td>
<td></td>
</tr>
<tr>
<td>Density</td>
<td>( \frac{</td>
<td>\dot{S}</td>
</tr>
</tbody>
</table>
Transmission line

- Transmission line = a pair of (twisted) cables used to transmit a signal
  - Current flows in one direction on one cable and comes back on the other cable
- If terminated correctly, Z is purely real: $Z = R_{\text{termination}}$
- Find $R$ when capacitance per unit length $= C'$ and inductance per unit length $= L'$
  - In theory:
    
    - In practice infinite sum of infinitesimal elements $C$ and $L$:

  
  *\[
  Z_\infty = \frac{1}{1 + i\omega C} \\left( \frac{1}{1 + i\omega L} \right)^2 = \frac{i\omega L}{1 + i\omega RC} + \frac{R}{1 + i\omega RC} = R
  \]

  *\[
  i\omega L - \omega^2 LCR + R = R + i\omega CR^2. \text{ Ignoring term with } LC \text{ (small): } \implies R = \frac{C}{\sqrt{L/C^2}}
  \]

Transmission line (2)

- What happens when transmission line is terminated correctly?
- $Z$ is purely real: $Z \approx R_{\text{termination}} \implies Z$ is a constant of the cable:
  - $Z$ does not depend on how long the cable is!
- If $R = \sqrt{L/C}$:
  - $Z$ will depend on how long the cable is and on the frequency of the signal
  - Distortions of the signal!
- Example of transmission line: coaxial cable, a pair of conducting tubes nested in one another
  - Homework: prove that for a cylindrical coaxial cable $Z = 2 \ln(b/a)/c$ and the velocity of propagation is $c$.
- Typical $R_{\text{termination}}$: 50 Ohm
Transmission line: demo

- Coaxial cable (127.4 m long)
- Pulse generator: pulse duration 0.1 µsec, period 20 µsec

Simultaneously send pulse from pulse generator (splitter)
- to Ch 1 of scope
- to transmission line (back and forth and display on Ch 2)
- Measure speed of propagation: Time difference: 656 ns \( \Rightarrow v=L/T\approx2/3c \)
- What happens if:
  - Open: signal will bounce back but nasty reflections
  - Short: signal will be reversed on the same cable, nothing on the other cable
  - If I terminate it with 50Ω resistor: signal comes back on return cable with no reflections

G. Sciolla – MIT 8.022 – Lecture 21

Scattering of light

(Logically this topic belongs to last lecture, but we did not have time...)

- When we send light into a medium, the light is scattered in many directions
- Example: light from Sun (unpolarized) passing through atmosphere
  - Propagation of light //z
  - We look up in x direction
  - What kind of light do we see?

G. Sciolla – MIT 8.022 – Lecture 21
Scattering of light (2)

- Since light propagates //z: no polarization // z
- We measure the light (with our eyes!) along the x direction: no polarization // x
  → The light we see must be polarized along the y direction
- This is actually not really true because the light scatters multiple times, but it suggests the general tendency
- What if the put a giant polaroid in front of the Sun?
  - Scattered light would be more intense in direction perpendicular to polarization direction
  - Rotating the polaroid would allow us to change intensity of the light:
    - Max intensity when polarization direction is // y axis
    - Dark when polarization direction is // x axis

Scattering of light (3)

- How is light scattered?
  - Light hits a molecule; the E shakes the molecule’s charges with frequency \( \omega \); the molecule re-radiates the light often changing the direction → changes in polarization
- Are all frequencies scattered in the same way?
  - Electric fields of scattered radiation depend on acceleration of (dipole) charges
    \[ E_{\text{scattered}} \propto \frac{\partial^2 \mathbf{d}}{\partial t^2} \propto \omega^2 \] if dipole moment of the shaken molecule goes as \( d \sim \cos \omega t \)
  - Intensity of scattered radiation: \( I \propto E_{\text{scattered}}^2 \propto \omega^4 \propto \lambda^{-4} \)
  - Since \( \lambda_{\text{red}} \approx 2 \lambda_{\text{blue}} \) → Blue is scattered 16 times more than red
    - This explains why the sky is blue during the day and why it’s red at sunset
Summary and outlook

Today:
- Energy and momentum carried by EM waves
  - Poynting vector and some of their applications
- Transmission lines
- Scattering of light
  - What happens at sunset?

Next Thursday:
- Magnetic fields through matter? Or review problems?

Sunset experiment

- Solution of distilled water and salt.
  - Unpolarized light is shining through it to the wall
- Add Na₂S₂O₃·35H₂O (Na thiosulfate)
  - Lights starts scattering: fog; light on the wall becomes red first and then dark as all the light is scattered toward the audience (as in sunset)
  - What happened?
    - Chemical reaction creates bigger and bigger molecules that scatter more and more light. Blue light is scattered first. Red makes it for a while but eventually scatters too.
- NB: light is polarized!
Sugar solution experiment (T8)

- Light goes through a polarizer and then through an optically active sugar solution.
- The first polarizer creates a linearly polarized wave, overlap of right-handed and left-handed circularly polarized waves which propagate at different speeds in the solution. This causes linear polarization direction to change slowly. Since the effect depends on $\lambda$, different colors are rotated differently.
- The second polarizer checks polarization direction at exit.