8.022 (E&M) – Lecture 3

Topics:
- Electric potential
- Energy associated with an electric field
- Gauss’s law in differential form

... and a lot of vector calculus... (yes, again!)

Last time...

What did we learn?
- Energy of a system of charges
  \[ U = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \frac{q_i q_j}{r_{ij}} \]
- Electric field
  \[ \vec{E} = \frac{\vec{F}}{q} = \frac{Q}{|r|^2} \hat{r} \]
- Gauss’s law in integral form:
  \[ \Phi = \oint_{S} \vec{E} \cdot d\vec{A} = 4\pi Q_{\text{enc}} \]
  Derived last time, but not rigorously...
Gauss’s law

- Consider charge in a generic surface $S$
- Surround charge with spherical surface $S_1$ concentric to charge
- Consider cone of solid angle $d\Omega$ from charge to surface $S$ through the little sphere
- Electric flux through little sphere:
  $$d\Phi_{s1} = \vec{E} \cdot d\vec{A} = (\frac{q}{r^2} \hat{r})(r^2d\Omega\hat{r}) = qd\Omega$$
- Electric flux through surface $S$:
  $$d\Phi_S = \vec{E} \cdot d\vec{A} = (\frac{q}{R^2} \hat{r})(R^2d\Omega\hat{r}) = qd\Omega$$
- $d\Phi_S = d\Phi_{s1} \Rightarrow \Phi_S = \Phi_{s1} = 4\pi Q$

$$\Phi = \oint_S \vec{E} \cdot d\vec{A} = 4\pi Q_{encl}$$ is valid for ANY shape $S$.

Confirmation of Gauss’s law

Electric field of spherical shell of charges:

$$\vec{E} = \begin{cases} \frac{Q}{r^2} \hat{r} & \text{outside the shell} \\ 0 & \text{inside the shell} \end{cases}$$

Can we verify this experimentally?

- Charge a spherical surface with Van de Graaf generator
- Is it charged? (D7 and D8)
- Is Electric Field radial?
  Does $E \sim 1/r^2$, eg: $\phi \sim 1/r$?
  Neon tube on only when oriented radially (D24)
- (D29?)
Confirmation of Gauss's law (2)

- Cylindrical shell positively charged
  - Gauss tells us that
    - $E_{\text{inside}} = 0$
    - $E_{\text{outside}} > 0$

- Can we verify this experimentally?
  - Demo D26
    - Charge 2 conductive spheres by induction outside the cylinder: one sphere will be + and the other will be −; it works because $E_{\text{outside}} > 0$
    - Try to do the same inside: nothing happens because $E = 0$

(Explain induction on the board)

Energy stored in E:

Squeezing charges...

- Consider a spherical shell of charge of radius $r$
- How much work $dW$ to “squeeze” it to a radius $r \cdot dr$?
- Guess the pressure necessary to squeeze it:
  \[
  P = \frac{F}{A} = \frac{QE}{A} = \frac{Q}{A} \sigma = E \sigma
  \]
  \[
  E_{\text{outside}} = \frac{Q}{r^2} \quad E_{\text{inside}} = 0 \quad \Rightarrow \quad E_{\text{surface}} = \frac{1}{2} \frac{Q}{r^2}
  \]
  \[
  \rightarrow P = E \sigma = \frac{1}{2} \frac{Q}{r^2} \sigma = \frac{\sigma}{2r^2} (4\pi r^3 \sigma) = 2\pi \sigma^2
  \]

- We can now calculate $dW$:
  \[
  dW = Fdr \quad (PA)dr = (2\pi \sigma^2)(4\pi r^3)dr = 2\pi \sigma^2 dV
  \]
  (where $dV = 4\pi r^2 dr$)

Remembering that $E_{\text{created in dr}} = 4\pi \sigma$ \quad $\Rightarrow$ \quad $dW = \frac{E^2}{8\pi} dV$
Energy stored in the electric field

- Work done on the system: \( dW = \frac{E^2}{8\pi} dV \)
  - We do work on the system (dW): same sign charges have been squeezed on a smaller surface, closer together and they do not like that...
- Where does the energy go?
  - We created electric field where there was none (between \( r \) and \( r-\text{dr} \))
    - The electric field we created must be storing the energy
  - Energy is conserved \( \rightarrow dU = dW \)
- \( U = \frac{E^2}{8\pi} \) is the energy density of the electric field \( E \)
- Energy is stored in the \( E \) field:
  - \( U = \int_{\text{Entire space}} \frac{E^2}{8\pi} dV \)
- NB: integrate over entire space not only where charges are!
  - Example: charged sphere

Electric potential difference

- Work to move \( q \) from \( r_1 \) to \( r_2 \):
  \[ W_{12} = \int_1^2 \vec{F}_{\text{Coulomb}} \cdot d\vec{s} = -\int_1^2 \vec{E}_{\text{Coulomb}} \cdot d\vec{s} = -q \int_1^2 \vec{E} \cdot d\vec{s} \]
- \( W_{12} \) depends on the test charge \( q \)
  - define a quantity that is independent of \( q \) and just describes the properties of the space:
  \[ \phi_{12} = \frac{W_{12}}{q} = -\int_1^2 \vec{E} \cdot d\vec{s} \]
- Physical interpretation:
  - \( \phi_{12} \) is work that I must do to move a unit charge from \( P_1 \) to \( P_2 \)
- Units:
  - cgs: statvolts = erg/esu; SI: Volt = N/C; 1 statvolts = \( 3 \times 10^7 \) V
Electric potential

- The electric potential difference $\phi_{12}$ is defined as the work to move a unit charge between $P_1$ and $P_2$: we need 2 points!
- Can we define similar concept describing the properties of the space?
  - Yes, just fix one of the points (e.g.: $P_1=$infinity):
    \[ \phi(\vec{r}) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{s} \quad \Leftarrow \text{Potential} \]
- **Application 1**: Calculate $\phi(\vec{r})$ created by a point charge in the origin:
  \[ \phi(\vec{r}) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{s} = -\int_{\infty}^{r} \frac{q}{r^2} \, dr = \frac{q}{r} \]
- **Application 2**: Calculate potential difference between points $P_1$ and $P_2$:
  \[ \phi_{12} = \int_{r_1}^{r_2} \vec{E} \cdot d\vec{s} = \frac{q}{r_2} - \frac{q}{r_1} = \phi(P_2) - \phi(P_1) \]
  → Potential difference is really the difference of potentials!

Potentials of standard charge distributions

The potential created by a point charge is $\phi(\vec{r}) = \frac{q}{r}$
  → Given this + superposition we can calculate anything!

- Potential of N point charges: $\phi(\vec{r}) = \sum_{i=1}^{N} \frac{q_i}{r_i}$
- Potential of charges in a volume $V$: $\phi(\vec{r}) = \int_{V} \frac{\rho \, dV}{r}$
- Potential of charges on a surface $S$: $\phi(\vec{r}) = \int_{S} \frac{\sigma \, dA}{r}$
- Potential of charges on a line $L$: $\phi(\vec{r}) = \int_{L} \frac{\lambda \, dl}{r}$
Some thoughts on potential

- **Why is potential useful? Isn’t E good enough?**
  - Potential is a scalar function \(\phi\) much easier to integrate than electric field or force that are vector functions.

- **When is the potential defined?**
  - Unless you set your reference somehow, the potential has no meaning.
  - Usually we choose \(\phi(\text{infinity})=0\).
    - This does not work always: e.g.: potential created by a line of charges.

- **Careful: do not confuse potential \(\phi(x,y,z)\) with potential energy of a system of charges \((U)\)**
  - Potential energy of a system of charges: \(U = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} q_i q_j / r_{ij}\)
  - Potential: work to move test charge from infinity to \((x,y,z)\) \(\phi(\vec{r}) = \sum_{i=1}^{N} q_i / r_i\).
Connection between $\phi$ and $E$

Consider potential difference between a point at $r$ and $r+dr$:

$$d\phi = -\int_{r}^{r+dr} \vec{E} \cdot d\vec{s} = -\vec{E}(\vec{r}) \cdot d\vec{r}$$

The infinitesimal change in potential can be written as:

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \cdot (dx, dy, dz) = \nabla \phi \cdot d\vec{r}$$

$$\vec{E} = -\nabla \phi$$

Useful info because it allows us to find $E$ given $\phi$
- Good because $\phi$ is much easier to calculate than $E$

Getting familiar with gradients...

1d problem:

$$\nabla f(x) = \frac{\partial f}{\partial x} \hat{x}$$

- The derivative $df/dx$ describes the function's slope
- The gradient describes the change of the function and the direction of the change

2d problem:

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

- The interpretation is the same, but in both directions
- The gradient points in the direction where the slope is deepest
Given the potential $\phi(x,y) = \sin(x)\sin(y)$, calculate its gradient.

$$\nabla \phi(x,y) = \cos(x)\sin(y)\hat{x} + \sin(x)\cos(y)\hat{y}$$

Visualization of gradients:

The gradient always points uphill $\rightarrow$ $E = -\nabla \phi$ points downhill

Visualization of gradients:

equipotential surfaces

Same potential $\phi(x,y) = \sin(x)\sin(y)$

NB: since equipotential lines are perpendicular to the gradient
$\rightarrow$ equipotential lines are always perpendicular to $E$
Divergence in E&M (1)

Consider flux of E through surface S:

\[ \Phi = \int_S \vec{E} \cdot d\vec{A} = \int_{S_1} \vec{E} \cdot d\vec{A} + \int_{S_2} \vec{E} \cdot d\vec{A} \]

Cut S into 2 surfaces: \( S_1 \) and \( S_2 \), with \( S_{\text{new}} \) the little surface in between

\[ \Phi = \int_{S_1} \vec{E} \cdot d\vec{A} + \int_{S_{\text{new}}} \vec{E} \cdot d\vec{A} + \int_{S_2} \vec{E} \cdot d\vec{A} \]

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Divergence Theorem

- Let's continue splitting into smaller volumes

\[ \Phi = \sum_{i=1}^{\text{large } N} \Phi_i = \sum_{i=1}^{\text{large } N} \int_{S_i} \vec{E} \cdot d\vec{A}_i = \sum_{i=1}^{\text{large } N} \frac{\int_{S_i} \vec{E} \cdot d\vec{A}_i}{V_i} \]

- If we define the divergence of E as

\[ \nabla \cdot \vec{E} \equiv \lim_{V \to 0} \frac{\int_{S_i} \vec{E} \cdot d\vec{A}}{V} \]

\[ \Phi = \sum_{i=4}^{\text{large } N} V_i \nabla \cdot \vec{E} \rightarrow \int \nabla \cdot \vec{E} dV \]

\[ \int_S \vec{E} \cdot d\vec{A} = \int_V \nabla \cdot \vec{E} dV \]

Divergence Theorem (Gauss's Theorem)
Gauss’s law in differential form

Simple application of the divergence theorem:

\[
\int_S \vec{E} \cdot d\vec{A} = \int_V \nabla \cdot \vec{E} \, dV
\]

\[
\int_S \vec{E} \cdot d\vec{A} = 4\pi Q = 4\pi \int_V \rho \, dV \rightarrow \int_V (\nabla \cdot \vec{E} - 4\pi \rho) \, dV = 0
\]

This is valid for any surface \( V \):

\[\nabla \cdot \vec{E} = 4\pi \rho\]

Comments:

- First Maxwell's equations
- Given \( E \), allows to easily extract charge distribution \( \rho \)

What’s a divergence?

- Consider infinitesimal cube centered at \( P=(x,y,z) \)
- Flux of \( F \) through the cube in \( z \) direction:

\[
\Delta \Phi_z = \int_{\text{cube}} \vec{F} \cdot d\vec{A} \Delta \Phi_z = \Delta \Phi_x = \Delta \Phi_y = \lim_{\Delta \to 0} \frac{1}{\Delta z}[F_z(x,y,z + \frac{\Delta z}{2}) - F_z(x,y,z - \frac{\Delta z}{2})]
\]

- Since \( \Delta z \to 0 \)

\[
\Delta \Phi_z = (\Delta x \Delta y \Delta z) \lim_{\Delta z \to 0} \frac{1}{\Delta z}[F_z(x,y,z + \frac{\Delta z}{2}) - F_z(x,y,z - \frac{\Delta z}{2})] = \Delta x \Delta y \Delta z \frac{\partial F_z}{\partial z}
\]

- Similarly for \( \Phi_x \) and \( \Phi_y \)

\[
\Delta \Phi_x = \Delta x \Delta y \Delta z \frac{\partial F_x}{\partial x} \quad \text{and} \quad \Delta \Phi_y = \Delta x \Delta y \Delta z \frac{\partial F_y}{\partial y}
\]
Divergence in cartesian coordinates

We defined divergence as

\[ \nabla \cdot \vec{F} = \lim_{\Delta \to 0} \frac{\oint_\Delta \vec{F} \cdot d\vec{A}}{V} \]

But what does this really mean?

\[ \nabla \cdot \vec{F} = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \lim_{\Delta z \to 0} \frac{\Delta x \Delta y \Delta z \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right)}{\Delta x \Delta y \Delta z} \]

\[ = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \]

This is the usable expression for the divergence: easy to calculate!

Application of Gauss's law in differential form

Problem: given the electric field \( E(r) \), calculate the charge distribution that created it

\[ \vec{E}(r) = \begin{cases} \frac{4}{3} \pi Kr\hat{r} & \text{for } r < R \quad \text{and} \quad \vec{E}(r) = \frac{4\pi K}{3r^2}R\hat{r} & \text{for } r > R \end{cases} \]

Hint: what connects \( E \) and \( \rho \)? Gauss's law.

\[ \oint_\Sigma \vec{E} \cdot d\vec{A} = 4\pi Q_{\text{enc}} \quad \text{(integral form)} \]

\[ \nabla \cdot \vec{E} = 4\pi \rho \quad \text{(differential form)} \]

In cartesian coordinates:

\[ \nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \begin{cases} 4\pi K & \text{when } r < R \\ 0 & \text{when } r > R \end{cases} \]

\( \rightarrow \) Sphere of radius \( R \) with constant charge density \( K \)
Next time...

- Laplace and Poisson equations
- Curl and its use in Electrostatics
- Into to conductors (?)