8.022 (E&M) – Lecture 5

Topics:
- More on conductors... and many demos!
- Capacitors

Last time...

- **Curl:** \( \text{curl } \vec{F} = \nabla \times \vec{F} \)
  - **Stoke's theorem:** \[ \oint_{C} \vec{F} \cdot d\vec{s} = \int_{A} \text{curl } \vec{F} \cdot d\vec{A} \quad \Rightarrow \quad \nabla \times \vec{E} = 0 \]
- **Laplacian:** \[ \nabla^2 \phi = \nabla \cdot \nabla \phi \]
  - \( \nabla^2 \phi = -4\pi \rho \) (Poisson) — in vacuum \( \Rightarrow \nabla^2 \phi = 0 \) (Laplace)
- **Conductors**
  - Materials with free electrons (e.g. metals)
  - Properties:
    - Inside a conductor \( \vec{E} = 0 \)
    - \( E_{\text{surface}} = 4\pi \rho \)
    - Field lines perpendicular to the surface \( \Rightarrow \) surface is equipotential
- **Uniqueness Theorem**
  - Given \( \rho(\text{xyz}) \) and boundary conditions, the solution \( \phi(\text{xyz}) \) is unique
Charge distribution on a conductor

- Let's deposit a charge $Q$ on a tear drop-shaped conductor
- How will the charge distribute on the surface? Uniformly?

- Experimental answer: NO! (Demo D28)
  - $\sigma_{\text{tip}} >> \sigma_{\text{flat}}$
  - Important consequence
    - Although $\phi=\text{const}$, $E=4\pi\sigma$
    - $E_{\text{tip}} >> E_{\text{flat}}$
- Why?

Charge distribution on a conductor (2)

- Qualitative explanation
  - Consider 2 spherical conductors connected by conductive wire
  - Radii: $R_1$ and $R_2$ with $R_1 >> R_2$
  - Deposit a charge $Q$ on one of them
    - charge redistributes itself until $\phi=\text{constant}$

$$\begin{align*}
\phi_1 &= \frac{Q}{R_1} = \frac{Q}{R_2} = \phi_2 \\
E_1 &= \frac{Q}{R_1} = \frac{\phi_1}{R_1} \\
E_2 &= \frac{Q}{R_2} = \frac{\phi_2}{R_2}
\end{align*}$$

$$\begin{align*}
E_1 &= \frac{R_1}{R_2} \Rightarrow E_2 &= \frac{R_1}{R_2} \Rightarrow \sigma_1 R_1 = \sigma_2 R_2
\end{align*}$$

- Conclusion:
  - Electric field is stronger where curvature ($1/R$) is larger
- More experimental evidence: D29 (Lightning with Van der Graaf)
Shielding

We proved that in a hollow region inside a conductor $E=0$

- This is the principle of shielding
- Do we need a solid conductor or would a mesh do?
  - Demo D32 (Faraday's cage in Van der Graaf)
  - Is shielding perfect?

Application of Uniqueness Theorem:

Method of images

- What is the electric potential created by a point charge $+Q$ at a distance $y$ from an infinite conductive plane?

- Consider field lines:
  - Radial around the charge
  - Perpendicular to the surface conductor

- The point charge $+Q$ induces – charges on the conductor
Method of images

- Apply the uniqueness theorem
  - It does not matter how you find the potential $\phi$ as long as the boundary conditions are satisfied. The solution is unique.
  - In our case: on the conductor surface: $\phi=0$ and always perpendicular
- Can we find an easier configuration of charges that will create the same field lines above the conductor surface?
  - YES!
  - For this system of point charges we can calculate $\phi(x,y,z)$ anywhere
  - This is THE solution (uniqueness)
  - NB: we do not care what happens below the surface of the conductor: that is nor the region under study

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Capacitance

- Consider 2 conductors at a certain distance
- Deposit charge $+Q$ on one and $-Q$ on the other
  - They are conductors
  - Each surface is equipotential
- What is the $\Delta \phi$ between the 2?
  - Let’s try to calculate:
    \[ V = \phi_2 - \phi_1 = -\int E \cdot ds = Q \times \text{(constant depending on geometry)} \]
  - Caveat: $C$ is proportional to $Q$ only if there is enough $Q$, uniformly spread...
- Naming the proportionality constant $1/C$: $Q = CV$
- Definitions:
  - $C =$ capacitance of the system
  - Capacitor: system of 2 oppositely charged conductors
Units of capacitance

- Definition of capacitance:
  \[ Q = CV \quad \Rightarrow \quad C = \frac{Q}{V} \]

- Units:
  - SI: Farad (F) = Coulomb/Volt
  - cgs: \( \text{cm} = \text{esu}/(\text{esu/cm}) \)
  - Conversion: \( 1 \text{ cm} = 1.11 \times 10^{-12} \text{ F} \sim 1 \text{ pF} \)

- Remember:
  - 1 Coulomb is a BIG charge: 1 F is a BIG capacitance
  - Usual \( C \sim \text{pF} - \mu\text{F} \)

Simple capacitors:

Isolated Sphere

- Conductive sphere of radius \( R \) in \((0,0,0)\) with a charge \( Q \)
  - Review questions:
    - Where is the charge located?
      - Hollow sphere? Solid sphere? Why?
    - What is the \( E \) everywhere in space?
  - Is this a capacitor?
    - Yes! The second conductor is a virtual one: infinity
  - Calculate the capacitance:
    \[
    \begin{align*}
    V &= \phi_R - \phi_\infty = \frac{Q}{R} \\
    Q &= Q \\
    \Rightarrow \quad C_{\text{sphere}} &= R
    \end{align*}
    \]

- Capacitors are everywhere!
The prototypical capacitor:
Parallel plates

- Physical configuration:
  - 2 parallel plates, each of area A, at a distance d
  - NB: if \(d^2 << A\) \(\rightarrow\) infinite parallel planes
  - Deposit +Q on top plate and -Q on bottom plate

- Capacitance:

\[
V = \int_{\text{top}}^{\text{bottom}} \vec{E} \cdot d\vec{s} = \int_{\text{top}}^{\text{bottom}} (4\pi \sigma) \vec{n} \cdot d\vec{n} = 4\pi \left(\frac{Q}{A}\right) d \\
\Rightarrow \quad C = \frac{Q}{V} = \frac{A}{4\pi d}
\]

Parallel plates capacitor: discussion

- Parallel plates capacitor:

\[
C = \frac{Q}{V} = \frac{A}{4\pi d}
\]

- Observations:
  - C depends only on the geometry of the arrangement
  - As it should, not on Q deposited or V between the plates!
  - Electric field on surface of conductor: \(2\pi \sigma\) or \(4\pi \sigma???
  - Infinite plane of charges: \(2\pi \sigma\)
    - With \(\sigma = Q/A\)
  - Conductor surface: \(4\pi \sigma\)
    - With \(\sigma = Q/2A\)
  - No contradiction if \(\sigma\) correctly defined!
  - What is the E outside the capacitor? Zero!
More review questions:

E in Nested Spherical Shells

- **Configuration:**
  - 2 concentric spherical shells
  - Charge: +Q (-Q) on inner (outer) sphere

- **Calculate E in the following regions:**
  - \( r<R_1, \) \( R_1<r<R_2, \) \( r>R_2 \)

Gauss's law is the key.
- \( \Phi_E \) on spherical surface with \( r<R_1, \) \( Q_{enc}=0 \) \( \Rightarrow \) \( E=0 \)
- \( \Phi_E \) on spherical surface with \( r>R_2, \) \( Q_{enc}=+Q-Q=0 \) \( \Rightarrow \) \( E=0 \)
- \( \Phi_E \) on spherical surface with \( R_1<r<R_2, \) \( Q_{enc}=+Q \) \( \Rightarrow \) \( E \neq 0 \)

\[
\Phi_E = \int E \cdot d\vec{s} = E(4\pi r) = 4\pi Q \Rightarrow \frac{E}{r} = \frac{Q}{r^2}
\]

More capacitors:

Nested Spherical Shells

- **Same configuration:**
  - 2 concentric spherical shells
  - Charge: +Q (-Q) on inner (outer) sphere

- **Capacitance:**
  - Key: finding the potential difference \( V \)

\[
V = \phi_1 - \phi_2 = -\int_{R_1}^{R_2} E \cdot d\vec{s} = -\int_{R_2}^{R_1} \frac{Q}{r^2} dr = \frac{Q}{R_1} - \frac{Q}{R_2} \Rightarrow C = \frac{Q}{V} = \frac{R_1 R_2}{R_2 - R_1}
\]

- If \( R_2-R_1<d<<R_2\Rightarrow 0 \)

\[
C = \frac{R_1 R_2}{R_2 - R_1} \sim \frac{R_2}{d} = \frac{4\pi R_2^2}{4\pi d} = A_{sphere} = \frac{4\pi}{4\pi d} \text{ same as plane capacitor!}
\]
Energy stored in a capacitor

- Consider a capacitor with charge +/-q
- How much work is needed to bring a positive charge dq from the negative plate to the positive plate?
  - NB: we are charging the capacitor!

\[ dW = V(q) dq = \frac{q}{C} dq \]

- How much work is needed to charge the capacitor from scratch?

\[ W = \int_0^Q dW = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C} \]

- Energy stored in the capacitor:

\[ U = \frac{Q^2}{2C} = \frac{1}{2} CV^2 \]

- Is this result consistent with what we found earlier?
  
  Example: parallel plate capacitor

\[ U = \frac{1}{8\pi} E^2 dV = \frac{1}{8\pi} E^2 Ad = \frac{1}{8\pi} (4\pi\sigma)^2 Ad \frac{A}{A} = \frac{Q^2}{2} (4\pi\frac{d}{A}) = \frac{1}{2} CV^2 \]

Cylindrical Capacitor

- Concentric cylindrical shells with charge +/-Q. Calculate:
  - Electric Field in between plates
    
    \[ \text{r}<a \text{ and } r>b: \text{E}=0 \text{ (Gauss)} \]
    
    \[ \text{a}<r<\text{b}: \text{Gauss’s law on cylinder of radius } r: \text{E}(r)= \frac{2Q}{L} \frac{r}{r} \]
  
  - V between plates:
    
    \[ V=\frac{1}{\text{\frac{L}{E}}}=\frac{2Q}{L} \frac{dr}{r} = \frac{2Q}{L} \ln \frac{\text{b}}{\text{a}} \]
  
  - Capacitance C:
    
    \[ C = \frac{Q}{V} = \frac{L}{2 \ln \frac{\text{b}}{\text{a}}} \]
  
  - Calculate energy stored in capacitor:
    
    \[ U = \frac{1}{2} CV^2 = \frac{1}{2} \frac{L}{2 \ln \frac{\text{b}}{\text{a}}} \left( \frac{2Q}{L} \ln \frac{\text{b}}{\text{a}} \right)^2 = \frac{Q^2}{L} \ln \frac{\text{b}}{\text{a}} \]
Next time...

- More on capacitors
- Charges in motion: currents
- Some help to get ready for quiz #1?
  - Review of Electrostatics?