ESG 8.022 Fall 2006 Exam 3

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1 Useful Formulae

You may find some of the following formulae useful. Then again, you may not.

Maxwell’s Equations: \( \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \); \( \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \); \( \nabla \cdot \vec{B} = 0 \); \( \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \)

Lorentz Force Law: \( \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \)

Conservation Laws: \( \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{S} = -\vec{J} \cdot \vec{E} \)

Impedance: \( \vec{V} = \vec{Z} \vec{I} \); \( \vec{Z}_R = R \); \( \vec{Z}_C = \frac{1}{j \omega C} \); \( \vec{Z}_L = j \omega L \)

Admittance: \( Y = 1/\vec{Z} \)

Potentials: \( \vec{E} = -\nabla \vec{V} \); \( \vec{B} = \nabla \times \vec{A} \)

Energy Density: \( u_{em} = \frac{1}{2 \mu_0} \vec{B}^2 + \frac{\varepsilon_0}{2} \vec{E}^2 \)

Poynting Vector: \( \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \)

Maxwell Stress Tensor: \( T_{ij} = \varepsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \)

2 Short (and Sweet) Answer Questions

Do four of the following five problems.

a. Charges are flowing in the \( \hat{y} \) direction through a flat plate of a conductor in the \( x-y \) plane, in a magnetic field in the \( \hat{z} \) direction. The \( +\hat{x} \) side of the conductor is measured to have a higher potential than the \( -\hat{x} \) side. What is the sign of the charge carriers?

i: positive

ii: negative

iii: both

iv: cannot be determined
b. A cylindrical wire made of imperfect conductor is connected to the two terminals of a battery so that current flows through the wire. Which of the following is true?
   i: There is a Poynting flux in the wire and its direction is parallel to the current.
   ii: There is a Poynting flux in the wire and its direction is radially outward (away from the central axis of the wire)
   iii: There is a Poynting flux in the wire and its direction is radially inward.
   iv: There is no Poynting flux in the wire

c. Which of the following is most responsible for paramagnetism?
   i: Lenz’s law
   ii: Alignment of permanent dipoles
   iii: The Pauli exclusion principle
   iv: Superconductivity

d. The Maxwell stress tensor of a uniform electric field corresponds to which case? (Hint: Think of field lines—don’t stress about the tensor)
   i: Isotropic pressure (same in every direction)
   ii: Isotropic tension (negative pressure)
   iii: Pressure along the field lines and tension perpendicular to them
   iv: Tension along the field lines and pressure perpendicular to them

e. A circular disk of radius $R$ has uniform surface charge density $\sigma$ and rotates like a wheel about its central axis with angular velocity $\omega$. The magnetic field for $r >> R$ is given by which of the following expressions? (HINT: Do not solve by brute force. There is a shortcut.) (HINT 2: There are no magnetic monopoles in the universe.)

\[
\begin{align*}
   i: & \quad \frac{\mu_0 \sigma \omega R^2}{4\pi r^2} (\cos\theta \hat{r} - \sin\theta \hat{\theta}) \\
   ii: & \quad \frac{\mu_0 \sigma \omega R^4}{4\pi r^2} (\cos\theta \hat{r} - \sin\theta \hat{\theta}) \\
   iii: & \quad \frac{\mu_0 \sigma \omega R^2}{16r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta}) \\
   iv: & \quad \frac{\mu_0 \sigma \omega R^4}{16r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta})
\end{align*}
\]
3 Problem with Circuits

Consider the situation where $V_{out}$ is an open circuit (no current goes through there).

a. Given an input voltage $V_{in} = V_0 \cos(\omega t)$, what current passes through the resistor?

b. What is the ratio of the output voltage amplitude to the input voltage amplitude?

c. What frequency, $\omega$, should you drive the circuit (as input voltage), to obtain the maximum output voltage amplitude?

d. Electrical engineers call such a circuit a band pass filter. Why does this name make sense given your answers to the previous questions?
4 Displacing the Problem

A capacitor C with circular plates of radius b is charged to a voltage $V_0$. The space between the two plates is small compared to b so that we can safely ignore any fringing effects. At $t = 0$ the switch is closed and the capacitor discharges through the resistor R. In all the questions below give your answers in terms of C, b, $V_0$, R, t and any universal constants.

a. Give an expression for the charge $Q(t)$ as a function of time of the positively charged plate (upper one in the figure) of the capacitor.

b. Find the electric field, $\vec{E}(t)$, between the capacitor plates.

c. Find the Maxwell displacement current density, $\vec{J}_d(t)$ between the two capacitor plates.

d. Find the magnetic field, $\vec{B}(t)$, between the capacitor plates. (Hint: Do not assume it is uniform)

e. Find the Poynting vector, $\vec{S}(t)$, between the capacitor plates.

f. Extra Credit: Calculate the time rate of change of the energy stored in the fields between the plates

g. : Extra Credit: Find a relation between your answers from the two previous parts. Comment on why this relation exists.