A. Line Integrals

The line integral of a scalar function \( f(x, y, z) \) along a path \( C \) is defined as

\[
\int_C f(x, y, z) \, ds = \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i, y_i, z_i) \Delta s_i
\]

where \( C \) has been subdivided into \( N \) segments, each with a length \( \Delta s_i \). To evaluate the line integral, it is convenient to parameterize \( C \) in terms of the arc length parameter \( s \). With \( x = x(s) \), \( y = y(s) \) and \( z = z(s) \), the above line integral can be rewritten as an ordinary definite integral:

\[
\int_C f(x, y, z) \, ds = \int_{s_1}^{s_2} f[x(s), y(s), z(s)] \, ds
\]

**Example 1:**

As an example, let us consider the following integral in two dimensions:

\[
I = \int_C (x + y) \, ds
\]

where \( C \) is a straight line from the origin to \((1,1)\), as shown in the figure. Let \( s \) be the arc length measured from the origin. We then have

\[
x = s \cos \theta = \frac{s}{\sqrt{2}}
\]

\[
y = s \sin \theta = \frac{s}{\sqrt{2}}
\]

The endpoint \((1,1)\) corresponds to \( s = \sqrt{2} \). Thus, the line integral becomes

\[
I = \int_0^{\sqrt{2}} \left( \frac{s}{\sqrt{2}} + \frac{s}{\sqrt{2}} \right) \, ds = \sqrt{2} \int_0^{\sqrt{2}} s \, ds = \sqrt{2} \cdot \frac{s^2}{2} \bigg|_0^{\sqrt{2}} = \sqrt{2}
\]
PROBLEM 1: *(Answer on the tear-sheet at the end!)*

In this problem, we would like to integrate the same function \( x + y \) as in Example 1, but along a different curve \( C' = C_1 + C_2 \), as shown in the figure. The integral can be divided into two parts:

\[
I' = \int_{C} (x + y) \, ds = \int_{C_1} (x + y) \, ds + \int_{C_2} (x + y) \, ds
\]

(a) Evaluate \( I_1 = \int_{C_1} (x + y) \, ds \).

(b) Evaluate \( I_2 = \int_{C_2} (x + y) \, ds \).

(c) Now add up \( I_1 \) and \( I_2 \) to obtain \( I' \). Is the value of \( I' \) equal to \( I = \sqrt{2} \) in Example 1 above? What can you conclude about the value of a line integral? That is, is the integral independent of the path you take to get from the beginning point to the end point?
B. Line Integrals involving Vector Functions

For a vector function
\[ \mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \]
the line integral along a path \( C \) is given by
\[ \int_C \mathbf{F} \cdot d\mathbf{s} = \int_C (F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}) \cdot (dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}) = \int_C F_x dx + F_y dy + F_z dz \]
where
\[ d\mathbf{s} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} \]
is the differential line element along \( C \). If \( \mathbf{F} \) represents a force vector, then this line integral is the work done by the force to move an object along the path.

PROBLEM 2: (Answer on the tear-sheet at the end!)

Let us evaluate the line integral of
\[ \mathbf{F}(x, y) = y \mathbf{i} - x \mathbf{j} \]
along the closed triangular path shown in the figure. Again, we divide the path into three segments \( C_1 \), \( C_2 \) and \( C_3 \), and evaluate the contributions separately. We will do the integral along \( C_i \) for you, as follows. Along \( C_1 \), the value of \( y \) is fixed at \( y = 0 \). With \( d\mathbf{s} = dx \mathbf{i} \), we have
\[ \mathbf{F}(x, 0) \cdot d\mathbf{s} = (-x \mathbf{j}) \cdot (dx \mathbf{i}) = 0 \]
So the integral along \( C_1 \) is zero. Now you will evaluate the integral along \( C_3 \). The value of \( x \) is fixed at \( x = 0 \), \( d\mathbf{s} = dy \mathbf{j} \), and \( \mathbf{F}(0, y) \cdot d\mathbf{s} = ? \)

(a) Evaluate \( \int_{C_1} \mathbf{F} \cdot d\mathbf{s} \).
Finally we calculate the contribution to the line integral from \( C_2 \). To evaluate the integral, we again parameterize \( x \) and \( y \) in terms of the arc length \( s \), which we take to be the distance between a point along \( C_2 \) and \((1,0)\). From the figure shown on the right, we have

\[
\frac{1-x}{s} = \cos 45^\circ = \frac{1}{\sqrt{2}}, \quad \frac{y}{s} = \sin 45^\circ = \frac{1}{\sqrt{2}}
\]

\[
x = 1 - \frac{s}{\sqrt{2}}, \quad y = \frac{s}{\sqrt{2}}
\]

and \( dx = -\frac{ds}{\sqrt{2}} \) and \( dy = \frac{ds}{\sqrt{2}} \).

(b) With the information given above, evaluate \( \int_{C_2} \vec{F} \cdot d\vec{s} \).

\[
F_x dx + F_y dy = ?
\]

\[
\int_C \vec{F} \cdot d\vec{s} = \int_{C_2} F_x dx + F_y dy = ?
\]
C. Surface Integrals

Double Integrals

A function $F(x, y)$ of two variables can be integrated over a surface $S$, and the result is a double integral:

$$\int\int_S F(x, y) dA = \int\int_S F(x, y) dx \, dy$$

where $dA = dx \, dy$ is a (Cartesian) differential area element on $S$. In particular, when $F(x, y) = 1$, we obtain the area of the surface $S$:

$$A = \int\int_S dA = \int\int_S dx \, dy$$

For example, the area of a rectangle of length $a$ and width $b$ (see figure) is simply given by

$$A = \int_0^b \int_0^a dx \, dy = \int_0^b \left( \int_0^a dx \right) dy \quad = \int_0^a dy = ab$$

Now suppose $F(x, y) = \sigma(x, y)$, where $\sigma$ is the charge density (Coulomb/m$^2$). Then the double integral represents the total charge on the surface:

$$Q = \int\int_S \sigma(x, y) dA = \int\int_S \sigma(x, y) dx \, dy$$

On the other hand, if the surface is a circle, it would be more convenient to work in polar coordinates.

The differential area element is given by (see figure above)
\[ dA = r \, dr \, d\theta \]

Integrating over \( r \) and \( \theta \), the area of a circle of radius \( R \) is

\[
A = \int_0^R \int_0^{2\pi} r \, d\theta \, dr = \int_0^R \left( \int_0^{2\pi} d\theta \right) r \, dr = \int_0^R 2\pi r \, dr = 2\pi \cdot \frac{R^2}{2} = \pi R^2
\]

as expected. If \( \sigma(r, \theta) \) is the charge distribution on a circular plate, then the total charge on the plate would be

\[
Q = \int_S \sigma(r, \theta) \, dA = \int_S \sigma(r, \theta) r \, dr \, d\theta
\]

**Closed Surface**

The surfaces we have discussed so far (rectangle and circle) are open surfaces. A closed surface is a surface which completely encloses a volume. An example of a closed surface is a sphere. To calculate the surface area of a sphere of radius \( R \), it is convenient to use spherical coordinates. The differential surface area element on the sphere is given by

\[
dA = R^2 \sin \theta \, d\theta \, d\phi
\]

Integrating over the polar angle \((0 \leq \theta \leq \pi)\) and the azimuthal angle \((0 \leq \phi \leq 2\pi)\), we obtain

\[
A = \iiint_S dA = \int_S R^2 \sin \theta \, d\theta \, d\phi
= R^2 \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\phi
= 4\pi R^2
\]

Suppose charge is uniformly distributed on the surface of the sphere of radius \( R \), then the total charge on the surface is

\[
Q = \iiint_S \sigma dA = 4\pi R^2 \sigma
\]

where \( \sigma \) is the charge density.
PROBLEM 3: *(Answer on the tear-sheet at the end!)*

(a) Find the total charge $Q$ on the rectangular surface of length $a$ ($x$ direction from $x = 0$ to $x = a$) and width $b$ ($y$ direction from $y = 0$ to $y = b$), if the charge density is $\sigma(x, y) = kxy$, where $k$ is a constant.

(b) Find the total charge on a circular plate of radius $R$ if the charge distribution is $\sigma(r, \theta) = kr(1 - \sin \theta)$. 
D. Surface Integrals involving Vector Functions

For a vector function \( \mathbf{F}(x, y, z) \), the integral over a surface \( S \) is given by

\[
\iint_S \mathbf{F} \cdot d\mathbf{A} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_S F_n \, dA
\]

where \( d\mathbf{A} = dA \mathbf{n} \) and \( \mathbf{n} \) is a unit vector pointing in the normal direction of the surface. The dot product \( F_n = \mathbf{F} \cdot \mathbf{n} \) is the component of \( \mathbf{F} \) parallel to \( \mathbf{n} \). The above quantity is called “flux.” For an electric field \( \mathbf{E} \), the electric flux through a surface is

\[
\Phi_\mathbf{E} = \iint_S \mathbf{E} \cdot \mathbf{n} \, dA = \iint_S E_n \, dA
\]

As an example, consider a uniform electric field \( \mathbf{E} = a\hat{i} + b\hat{j} \) which intersects a surface of area \( A \). What is the electric flux through this area if the surface lies in the \( yz \) plane with normal in the positive \( x \) direction? In this case, the normal vector is \( \mathbf{n} = \hat{i} \), pointing in the \( +x \) direction. The electric flux through this surface is

\[
\Phi_\mathbf{E} = \mathbf{E} \cdot \mathbf{A} = \left(a\hat{i} + b\hat{j}\right) \cdot A\hat{i} = aA
\]

**PROBLEM 4:** *(Answer on the tear-sheet at the end!)*

(a) Consider a uniform electric field \( \mathbf{E} = a\hat{i} + b\hat{j} \) which intersects a surface of area \( A \). What is the electric flux through this area if the surface lies (i) in the \( xz \) plane with normal in the positive \( y \) direction? (ii) in the \( xy \) plane with the normal in the positive \( z \) direction?
A cylinder has base radius $R$ and height $h$ with its axis along the $z$-direction. A uniform field $\mathbf{E} = E_0 \mathbf{j}$ penetrates the cylinder. Determine the electric flux $\iint_S \mathbf{E} \cdot \mathbf{n} \, dA$ for the side of the cylinder with $y > 0$, where the area normal points away from the interior of the cylinder.

Hints: If $\theta$ is the angle in the $xy$ plane measured from the $x$-axis toward the positive $y$-axis, what is the differential area of the side of the cylinder in terms of $R$, $dz$, and $d\theta$?

What is the vector formula for the normal $\mathbf{n}$ to the side of the cylinder with $y > 0$, in terms of $\theta$, $\mathbf{i}$ and $\mathbf{j}$? What is $\mathbf{E} \cdot \mathbf{n}$?

$$\iint_S \mathbf{E} \cdot \mathbf{n} \, dA = ?$$
PROBLEM 1:

(a) \( I_1 = \int_{C_1} (x + y) \, ds = \)

(b) \( I_2 = \int_{C_2} (x + y) \, ds = \)

(c) \( I' = I_1 + I_2 = \)

Is the value of \( I' \) equal to \( I = \sqrt{2} \) in Example 1 above? What can you conclude about the value of a line integral? That is, is the integral independent of the path you take to get from the beginning point to the end point?

PROBLEM 2:

(a) \( \int_{C_3} \mathbf{F} \cdot d\mathbf{s} = \)

(b) \( \int_{C_4} \mathbf{F} \cdot d\mathbf{s} = \)
PROBLEM 3:

(a) Total charge $Q =$

(b) Total charge $Q =$

PROBLEM 4:

(a) Consider a uniform electric field $\vec{E} = a \hat{i} + b \hat{j}$ which intersects a surface of area $A$. What is the electric flux through this area if the surface lies

(i) in the $xz$ plane?

(ii) in the $xy$ plane?

(b) Determine the electric flux $\oiint S \vec{E} \cdot \hat{n} dA$ for the side of the cylinder with $y > 0$. 