Problem Solving 8: Driven LRC Circuits

OBJECTIVES

1. To explore the relationship between driven current and driving emf in three simple circuits that contain: (1) only resistance; (2) only inductance; and (3) only capacitance.

2. To examine these same relationships in the general case where $R$, $L$, and $C$ are all present, and to do two sample problems on the LRC circuits.

REFERENCE: Sections 12.1 – 12.4, 8.02 Course Notes.

General Properties of Driven LRC Circuits

We have previously considered the “free” oscillations of an LRC circuit. These are the oscillations we see if we just “kick” the circuit and stand back and watch it oscillate. If we do this we will see a natural frequency of oscillation that decays in a finite time. Here we consider a very different problem. We now “drive” the LRC circuit with a source of emf with some (arbitrary) amplitude and frequency. If we drive the circuit with an $emf \ V(t) = V_0 \sin \omega t$, where $\omega$ is any frequency we desire (we get to pick this) and $V_0$ is any amplitude we desire, then the “driven” response of the system, as opposed to its natural “free” oscillations (which we assume have exponentially decayed to zero) is given by

$$I(t) = I_0 \sin(\omega t - \phi)$$

where

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\frac{\omega L - \frac{1}{\omega C}}{\omega L - \frac{1}{\omega C}}\right)^2}}, \quad \tan \phi = \left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

(8.1)

Note the “driven” response is at the (arbitrary) frequency of the driver, and not at any natural frequency of the system. However the system will show maximum response to the driving emf when the driving frequency is at the natural frequency of oscillation of the system, i.e. when $\omega = 1/\sqrt{LC}$. We can compute the average power consumed by the circuit by calculating the time average of $I(t)V(t)$ (see Section 12.4, 8.02 Course Notes):

$$\langle P(t) \rangle = \langle I(t)V(t) \rangle = \frac{1}{2} I_0 V_0 \cos \phi$$

(8.2)
**Example 1: Driven circuit with resistance only**

We begin with a circuit which contains only resistance. The circuit diagram is shown below.

![Circuit Diagram]

The circuit equation is

\[ I_R(t)R - V(t) = 0. \]

**Question 1:** What is the amplitude \( I_{R0} \) and phase \( \phi \) of the current \( I_R(t) = I_{R0} \sin(\omega t - \phi) \)?

**Answer:** *(answer this and subsequent questions on the tear-off sheet at the end!!)*

**Question 2:** What values of \( L \) and \( C \) do you choose in the general equation (8.1) to reproduce the result you obtained in your answer above?

**Answer:**

**Question 3:** What is the *time-averaged* power \( \langle P_R(t) \rangle = \langle I_R(t)V_R(t) \rangle \) dissipated in this circuit? You will need to know that the time average of \( \sin^2 \omega t \) is \( \langle \sin^2 \omega t \rangle = 1/2 \).

**Answer:**
Example 2: Driven circuit with inductance only

Now suppose the voltage source \( V(t) = V_0 \sin(\omega t) \) is connected in a circuit with only self-inductance. The circuit diagram is

\[
\begin{array}{c}
\text{Source}
\end{array}
\]

The circuit equation is

\[
V(t) = L \frac{dI}{dt}
\]

**Question 4:** Solve the above equation for the current as a function of time. If we write this current in the form \( I_L(t) = I_{L0} \sin(\omega t - \phi) \), what is the amplitude \( I_{L0} \) and phase \( \phi \) of the current? You will need to use the trigonometric identity

\[
\sin(\omega t - \phi) = \sin \omega t \cos \phi - \sin \phi \cos \omega t.
\]

**Answer:**

**Question 5:** What values of \( R \) and \( C \) do you choose in the general equation (8.1) to reproduce the result you obtained in the question above?

**Answer:**

**Question 6:** What is the time-averaged power \( \langle P_L(t) \rangle = \langle I_L(t)V_L(t) \rangle \) dissipated in this circuit? You will need to know that the time-average of \( \sin \omega t \cos \omega t \) is \( \langle \sin \omega t \cos \omega t \rangle = 0 \).

**Answer:**
Example 3: Driven circuit with capacitance only

The ac voltage source \( V(t) = V_0 \sin(\omega t) \) is connected in a circuit with capacitance only. The circuit diagram is

![Circuit Diagram](image)

\[ V = V_0 \sin \omega t \]

The circuit equation for this circuit is

\[
\frac{Q}{C} - V(t) = 0
\]

If we take the time derivative of this equation we get

\[
\frac{I_C}{C} - \frac{d}{dt} V(t) = \frac{I_C}{C} - \omega V_0 \cos \omega t = 0
\]

**Question 7:** Solve the above equation for the current as a function of time. If we write this current in the form \( I_C(t) = I_{c0} \sin(\omega t - \phi) \), what is the amplitude \( I_{c0} \) and phase \( \phi \) of the current? You will need to use the trigonometric identity \( \sin(\omega t - \phi) = \sin \omega t \cos \phi - \sin \phi \cos \omega t \).

**Answer:**

**Question 8:** What is the time-averaged power \( \langle P_C(t) \rangle = \langle I_C(t) V_C(t) \rangle \) dissipated in this circuit? You will need to know that the time-average of \( \sin \omega t \cos \omega t \) is \( \langle \sin \omega t \cos \omega t \rangle = 0 \).

**Answer:**

Friday 4/15/2005 Solving8-4
Sample Problem 1

The circuit shown below contains an AC generator which provides a source of sinusoidally varying emf $\varepsilon(t) = \varepsilon_0 \sin \omega t$, a resistor with resistance $R$, and a "black box", which contains either an inductor or a capacitor, but not both. The amplitude of the driving emf, $\varepsilon_0$, is 100 Volts/meter, and the angular frequency $\omega$ is 10 rad/sec. We measure the current in the circuit and find that it is given as a function of time by the expression: $I(t) = (10 \text{ Amps}) \sin(\omega t - \pi/4)$ [Note: $\pi/4$ radians = $45^\circ$, $\tan(\pi/4) = +1$].

**Question 9:** Does this current lead or lag the emf $\varepsilon(t) = \varepsilon_0 \sin(\omega t)$

**Answer:**

**Question 10:** What is the unknown circuit element in the black box--an inductor or a capacitor?

**Answer:**

**Question 11:** What is the numerical value of the resistance $R$? Your answer can contain square roots, if appropriate. Indicate units.

**Answer:**

**Question 12:** What is the numerical value of the capacitance or of the inductance, as the case may be? Your answer can contain square roots, if appropriate. Indicate units.

**Answer:**

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**Sample Problem 2**

The circuit shown below contains an AC generator which provides a source of sinusoidally varying emf $\varepsilon(t) = \varepsilon_0 \sin(\omega t)$, a resistor with resistance $R = 1$ ohm, and a "black box", which contains either an inductor or a capacitor, or both. The amplitude of the driving emf, $\varepsilon_0$, is 1 Volt. We measure the current in the circuit at an angular frequency $\omega = 1$ radians/sec and find that it is exactly in phase with the driving emf. We measure the current in the circuit at an angular frequency $\omega = 2$ radians/sec and find that it lags the driving emf by exactly $\pi/4$ radians. [Note: $\pi/4$ radians = $45^\circ$, $\tan(\pi/4) = 1$].

**Question 13:** What does the black box contain--an inductor or a capacitor, or both? Explain your reasoning.

**Answer:**

**Question 14:** What is the numerical value of the capacitance or of the inductance, or of both, as the case may be? Indicate units. Your answer(s) will involve simple fractions only, you will not need a calculator to find the value(s).

**Answer:**

**Question 15:** What is the numerical value of the time-averaged power dissipated in this circuit when $\omega = 1$ radians/sec? Indicate units, that is the time-average of $I(t)V(t)$. You will need to know that the time-average of $\sin^2 \omega t$ is $1/2$.

**Answer:**
Problem Solving 8: Driven RLC Circuits

Group _______________________________ (e.g. 6A Please Fill Out)

Names ______________________________

_______________________________

Example 1: Driven circuit with resistance only

Question 1: What is the amplitude $I_{R0}$ and phase $\phi$ of the current $I_R(t) = I_{R0} \sin(\omega t - \phi)$?

Answer: $I_{R0}$: ________________________  $\phi$: _________________

Question 2: What values of $L$ and $C$ do you choose in the general equation (8.1) to reproduce the result you obtained in your answer above?

Answer: $L$: ________________________  $C$: _________________

Question 3: What is the time-averaged power $\langle P_R(t) \rangle = \langle I_R(t) V_R(t) \rangle$ dissipated?

Answer: $\langle P_R(t) \rangle$ = ________________________

Example 2: Driven circuit with inductance only

Question 4: What is the amplitude $I_{L0}$ and phase $\phi$ of the current $I_L(t) = I_{L0} \sin(\omega t - \phi)$?

Answer: $I_{L0}$: ________________________  $\phi$: _________________

Question 5: What values of $R$ and $C$ do you choose in the general equation (8.1) to reproduce the result you obtained in the question above?

Answer: $R$: ________________________  $C$: _________________
**Question 6:** What is the *time-averaged* power \( \langle P_L(t) \rangle = \langle I_L(t) V_L(t) \rangle \) dissipated?

Answer: \( \langle P_L(t) \rangle = \text{________________________} \)

**Example 3: Driven circuit with capacitance only**

**Question 7:** What is the amplitude \( I_{C0} \) and phase \( \phi \) of the current \( I_c(t) = I_{C0} \sin(\omega t - \phi) \)?

Answer: \( I_{C0} : \text{________________________} \) \( \phi : \text{________________________} \)

**Question 8:** What is the *time-averaged* power \( \langle P_C(t) \rangle = \langle I_c(t) V_c(t) \rangle \) dissipated?

Answer: \( \langle P_C(t) \rangle = \text{________________________} \)

**Sample Problem 1:**

**Question 9:** Does this current lead or lag the emf \( \varepsilon(t) = \varepsilon_0 \sin \omega t \)?

Answer: ___________________

**Question 10:** What is the unknown circuit element in the black box--an inductor or a capacitor?

Answer: ___________________

**Question 11:** What is the numerical value of the resistance \( R \)? Your answer can contain square roots, if appropriate. Indicate units.

Answer: ___________________

**Question 12:** What is the numerical value of the capacitance or of the inductance, as the case may be? Your answer can contain square roots, if appropriate. Indicate units.

Answer: ___________________

**Sample Problem 2:**

**Question 13:** What does the black box contain--an inductor or a capacitor, or both? Explain your reasoning.

Answer: ___________________

**Question 14:** What is the numerical value of the capacitance or of the inductance, or of both, as the case may be? Indicate units. Your answer(s) will involve simple fractions only, you will not need a calculator to find the value(s).

Answer: \( L: \text{________________________} \) \( C: \text{________________________} \)

**Question 15:** What is numerical value of the *time-averaged* power dissipated in this circuit when \( \omega = 1 \) radians/sec? Indicate units.

Answer: ___________________