Class 26: Outline

Hour 1:
  Driven Harmonic Motion (RLC)

Hour 2:
  Experiment 11: Driven RLC Circuit
Last Time:
Undriven RLC Circuits
LC Circuit

It undergoes simple harmonic motion, just like a mass on a spring, with trade-off between charge on capacitor (Spring) and current in inductor (Mass)
Damped LC Oscillations

Resistor dissipates energy and system rings down over time.
Mass on a Spring: Simple Harmonic Motion ` A Second Look
Mass on a Spring

We solved this:

\[ F = -kx = ma = m \frac{d^2x}{dt^2} \]

\[ m \frac{d^2x}{dt^2} + kx = 0 \]

Simple Harmonic Motion

\[ x(t) = x_0 \cos(\omega_0 t + \phi) \]

Moves at natural frequency

What if we now move the wall? Push on the mass?
Demonstration:
Driven Mass on a Spring
Off Resonance
Driven Mass on a Spring

Now we get:

\[ F = F(t) - kx = ma = m \frac{d^2x}{dt^2} \]

\[ m \frac{d^2x}{dt^2} + kx = F(t) \]

Assume harmonic force:

\[ F(t) = F_0 \cos(\omega t) \]

Simple Harmonic Motion

\[ x(t) = x_{\text{max}} \cos(\omega t + \phi) \]

Moves at driven frequency
Resonance

\[ x(t) = x_{\text{max}} \cos(\omega t + \phi) \]

Now the amplitude, \( x_{\text{max}} \), depends on how close the drive frequency is to the natural frequency.

Let’s See…
Demonstration: Driven Mass on a Spring
Resonance

\[ x(t) = x_{\text{max}} \cos(\omega t + \phi) \]

\( x_{\text{max}} \) depends on drive frequency

Many systems behave like this:
- Swings
- Some cars
- Musical Instruments

...
Electronic Analog: RLC Circuits
Analog: RLC Circuit

Recall:
- Inductors are like masses (have inertia)
- Capacitors are like springs (store/release energy)
- Batteries supply external force (EMF)

Charge on capacitor is like position,
Current is like velocity – watch them resonate

Now we move to “frequency dependent batteries:”
AC Power Supplies/AC Function Generators
Demonstration: RLC with Light Bulb
Start at Beginning:
AC Circuits
Alternating-Current Circuit

- direct current (dc) – current flows one way (battery)
- alternating current (ac) – current oscillates

- sinusoidal voltage source

\[ V(t) = V_0 \sin \omega t \]

\[ \omega = 2\pi f : \text{angular frequency} \]

\[ V_0 : \text{voltage amplitude} \]
AC Circuit: Single Element

\[ V \quad =\quad V \]
\[ = V_0 \sin \omega t \]

\[ I(t) = I_0 \sin(\omega t - \phi) \]

Questions:
1. What is \( I_0 \) ?
2. What is \( \phi \) ?
AC Circuit: Resistors

\[ I_R = \frac{V_R}{R} = \frac{V_0}{R} \sin \omega t \]

\[ = I_0 \sin (\omega t - 0) \]

\[ I_0 = \frac{V_0}{R} \]

\[ \varphi = 0 \]

\[ V_R = I_R R \]

\[ V = V_0 \sin \omega t \]

\[ I_R \text{ and } V_R \text{ are in phase} \]
AC Circuit: Capacitors

\[ I_c(t) = \frac{dQ}{dt} = \omega CV_0 \cos \omega t = I_0 \sin(\omega t - \frac{\pi}{2}) \]

\[ I_0 = \omega CV_0 \]
\[ \varphi = -\frac{\pi}{2} \]

\[ Q(t) = CV_c = CV_0 \sin \omega t \]

\[ V = V_0 \sin \omega t \]

\[ V_c = \frac{Q}{C} \]

\[ I_c \text{ leads } V_c \text{ by } \pi/2 \]
AC Circuit: Inductors

\[ I_L(t) = V_0 / L \int \sin \omega t \, dt = -\frac{V_0}{\omega L} \cos \omega t = I_0 \sin (\omega t - \pi/2) \]

\[ I_0 = \frac{V_0}{\omega L} \]

\[ \phi = \frac{\pi}{2} \]

\( I_L \) lags \( V_L \) by \( \pi/2 \)

\[ V = V_0 \sin \omega t \]

\[ \frac{dI_L}{dt} = \frac{V_L}{L} = \frac{V_0}{L} \sin \omega t \]
## AC Circuits: Summary

<table>
<thead>
<tr>
<th>Element</th>
<th>$I_0$</th>
<th>Current vs. Voltage</th>
<th>Resistance Reactance Impedance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>$\frac{V_{0R}}{R}$</td>
<td>In Phase</td>
<td>$R = R$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$\omega C V_{0C}$</td>
<td>Leads</td>
<td>$X_C = \frac{1}{\omega C}$</td>
</tr>
<tr>
<td>Inductor</td>
<td>$\frac{V_{0L}}{\omega L}$</td>
<td>Lags</td>
<td>$X_L = \omega L$</td>
</tr>
</tbody>
</table>

Although derived from single element circuits, these relationships hold generally!
PRS Question: Leading or Lagging?
Phasor Diagram

Nice way of tracking magnitude & phase:

\[ V(t) = V_0 \sin(\omega t) \]

Notes: (1) As the phasor (red vector) rotates, the projection (pink vector) oscillates
(2) Do both for the current and the voltage
Demonstration: Phasors
Phasor Diagram: Resistor

$V_0 = I_0 R$

$\varphi = 0$

$I_R$ and $V_R$ are in phase
Phasor Diagram: Capacitor

$I_0$ leads $V_c$ by $\pi/2$

\[ V_0 = I_0 X_C \]
\[ = I_0 \frac{1}{\omega C} \]
\[ \varphi = -\frac{\pi}{2} \]
Phasor Diagram: Inductor

\[ V_0 = I_0 X_L \]
\[ = I_0 \omega L \]
\[ \varphi = \frac{\pi}{2} \]

**\( I_L \) lags \( V_L \) by \( \pi/2 \)**
PRS Questions: Phase
Put it all together:
Driven RLC Circuits
Question of Phase

We had fixed phase of voltage:

\[ V = V_0 \sin \omega t \quad I(t) = I_0 \sin (\omega t - \phi) \]

It’s the same to write:

\[ V = V_0 \sin(\omega t + \phi) \quad I(t) = I_0 \sin \omega t \]

(Just shifting zero of time)
Driven RLC Series Circuit

\[ I(t) = I_0 \sin(\omega t) \]

\[ V_R = V_{R0} \sin(\omega t) \]

\[ V_L = V_{L0} \sin(\omega t + \frac{\pi}{2}) \]

\[ V_C = V_{C0} \sin(\omega t + \frac{-\pi}{2}) \]

What is \( I_0 \) (and \( V_{R0} = I_0 R \), \( V_{L0} = I_0 X_L \), \( V_{C0} = I_0 X_C \))? What is \( \phi \)? Does the current lead or lag \( V_s \)?

Must Solve: \( V_s = V_R + V_L + V_C \)
Driven RLC Series Circuit

Now Solve: \( V_S = V_R + V_L + V_C \)

Now we just need to read the phasor diagram!
Driven RLC Series Circuit

\[ I(t) = I_0 \sin(\omega t - \varphi) \]

\[ V_S = V_{0S} \sin(\omega t) \]

\[
V_{0S} = \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2} = I_0 \sqrt{R^2 + (X_L - X_C)^2} = I_0 Z
\]

\[
I_0 = \frac{V_{0S}}{Z}
\]

\[
Z = \sqrt{R^2 + (X_L - X_C)^2}
\]

\[
\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)
\]

Impedance
Plot I, V’s vs. Time

\[ I(t) = I_0 \sin (\omega t) \]

\[ V_R(t) = I_0 R \sin (\omega t) \]

\[ V_L(t) = I_0 X_L \sin (\omega t + \frac{\pi}{2}) \]

\[ V_L(t) = I_0 X_C \sin (\omega t - \frac{\pi}{2}) \]

\[ V_S(t) = V_{S0} \sin (\omega t + \phi) \]
PRS Question: Who Dominates?
RLC Circuits: Resonances
At very low frequencies, C dominates ($X_C \gg X_L$): it fills up and keeps the current low.

At very high frequencies, L dominates ($X_L \gg X_C$): the current tries to change but it won’t let it.

At intermediate frequencies we have **resonance**

$I_0$ reaches maximum when $X_L = X_C$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$
Resonance

\[ I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C} \]

\[ \phi = \tan^{-1} \left( \frac{X_L - X_C}{R} \right) \]

C-like: \( \phi < 0 \)

I leads

L-like: \( \phi > 0 \)

I lags

\[ \Delta \omega = \frac{1}{\sqrt{LC}} \]
Demonstration: RLC with Light Bulb
PRS Questions: Resonance
Experiment 11: Driven RLC Circuit
Experiment 11: How To

Part I
• Use exp11a.ds
• Change frequency, look at I & V. Try to find resonance – place where I is maximum

Part II
• Use exp11b.ds
• Run the program at each of the listed frequencies to make a plot of $I_0$ vs. $\omega$