Electricity and Magnetism

• Reminder
  – Wave recap.
  – Maxwell’s Equations in differential form

• Today
  – From Maxwell’s Equations to E.M. waves
  – Properties of Electromagnetic waves
Electromagnetic Waves

• How did Maxwell predict electromagnetic waves?
  - What are waves?
  - Wave equation
  - Maxwell’s Equations in differential form
  - E.M. wave equation
  - Properties of E.M. waves

Today
Reminder on Waves

At a moment in time:

\[ D(X) \]

Wavelength \( \lambda \)

Amplitude

Position \( X \)

At a point in space:

\[ D(t) \]

Period \( T = 1/f \)

Amplitude

Time \( t \)

May 6  2002
Wave Equation

- Wave equation:

\[ \frac{\partial^2 D(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 D(x,t)}{\partial t^2} \]

Couples variation in time and space

- Speed of propagation: \( v = \lambda f \)

- *How can we derive a wave equation from Maxwell's equations?*
Wave properties

• What do we want to know about waves:
  – Speed of propagation?
  – Transverse or longitudinal oscillation?
  – What is oscillating?
  – What are typical frequencies/wavelengths?
Back to Maxwell’s equation

- Wave equation is differential equation
- M.E. (so far) describe integrals of fields

\[
\oint_{A_{\text{closed}}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon_0} \\
\oint_{L_{\text{closed}}} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \\
\oint_{A_{\text{closed}}} \vec{B} \cdot d\vec{A} = 0 \\
\oint_{L_{\text{closed}}} \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}
\]
Differential Form of M.E.

\begin{align*}
\oint_{A_{\text{closed}}} \vec{E} \cdot d\vec{A} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
\oint_{L_{\text{closed}}} \vec{E} \cdot d\vec{l} &= -\frac{d\Phi_B}{dt} \\
\oint_{A_{\text{closed}}} \vec{B} \cdot d\vec{A} &= 0 \\
\oint_{L_{\text{closed}}} \vec{B} \cdot d\vec{l} &= \mu_0 I_{\text{encl}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}
\end{align*}

\begin{align*}
\nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}
\end{align*}
Gauss Theorem

\[ \int_{V(A)} \nabla \cdot \vec{F} \, dV = \oint_{A} \vec{F} \cdot d\vec{A} = \Phi_F \]

Flux/Unit Volume

Divergence

\[ \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \]
Stokes Theorem

\[ \oint_L \vec{F} \, dl = \int_{A(L)} \nabla \times \vec{F} \, dA \]

\( \nabla \times \vec{F} = \vec{i} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \vec{j} \left( \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) + \vec{k} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \)
Differential Form of M.E.

\[
\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \\
\nabla \cdot \vec{B} = 0 \\
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \\
\]

Flux/Unit Volume

Charge Density

Loop Integral/Unit Area

Current Density
Q: Do we need \( \rho \) and \( \vec{j} \) to understand E.M. waves?

A: No! Light travels from sun to earth, i.e. in vacuum (no charge, no current)!

There’s no ‘medium’ involved!?  
- unlike waves on water or sound waves
Maxwell’s Equations in Vacuum

• Look at Maxwell’s Equations without charges, currents

\[ \nabla \cdot \vec{E} = 0 \]
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
\[ \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \]

Now completely symmetric!
Maxwell’s Equations in Vacuum

I. \( \nabla \cdot \vec{E} = 0 \)

II. \( \nabla \cdot \vec{B} = 0 \)

III. \( \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \)

VI. \( \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \)

Allow variations only in z-direction:

\[ \frac{\partial}{\partial x} = 0 \]
\[ \frac{\partial}{\partial y} = 0 \]
Illustration

2-D wave: $x, z, D(x, z, t)$

$$\frac{\partial}{\partial x} = 0$$
Maxwell’s Equations in Vacuum

I. \[ \nabla \cdot \vec{E} = 0 \]

II. \[ \nabla \cdot \vec{B} = 0 \]

III. \[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

VI. \[ \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \]

Allow variations only in z-direction:

\[ \frac{\partial}{\partial x} = 0 \]
\[ \frac{\partial}{\partial y} = 0 \]
Electromagnetic Waves

- We found wave equations:

\[
\begin{align*}
\frac{\partial^2 B_y}{\partial z^2} &= \frac{1}{c^2} \frac{\partial^2 B_y}{\partial t^2} \\
\frac{\partial^2 E_y}{\partial z^2} &= \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2}
\end{align*}
\]

same for \( E_x, B_x \)

\( v = c \)

E and B are oscillating!
Electromagnetic Waves

- Note: $(E_x, B_y)$ and $(E_y, B_x)$ independent:

\[
\begin{align*}
\frac{\partial B_x}{\partial z} &= \frac{1}{c^2} \frac{\partial E_y}{\partial t}
\frac{\partial B_y}{\partial z} &= -\frac{1}{c^2} \frac{\partial E_x}{\partial t}
\frac{\partial E_x}{\partial z} &= -\frac{\partial B_y}{\partial t}
\frac{\partial E_y}{\partial z} &= \frac{\partial B_x}{\partial t}
\end{align*}
\]
Plane waves

- Example solution: Plane waves

\[ E_y = E_0 \cos(kz - \omega t) \]
\[ B_x = B_0 \cos(kz - \omega t) \]

with \( k = \frac{2\pi}{\lambda}, \omega = 2\pi f \) and \( f\lambda = c \).

- We can express other functions as linear combinations of sin, cos
  - ‘White’ light is combination of waves of different frequency
  - In-Class Demo...
Plane waves

- Example solution: Plane waves

\[ E_y = E_0 \cos(kz - \omega t) \]
\[ B_x = B_0 \cos(kz - \omega t) \]

with \( k = \frac{2\pi}{\lambda}, \) \( \omega = 2\pi f \) and \( f \lambda = c. \)

Check

\[ \frac{\partial^2 B_y}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 B_y}{\partial t^2} \]
\[ \frac{\partial^2 E_y}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} \]

\[ \frac{\partial B_x}{\partial z} = \frac{1}{c^2} \frac{\partial E_y}{\partial t} \]
\[ \frac{\partial B_y}{\partial z} = -\frac{1}{c^2} \frac{\partial E_x}{\partial t} \]
\[ \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \]
\[ \frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t} \]

\[ -kE_0 \sin(kz - \omega t) = \omega B_0 \sin(kz - \omega t) \]

\[ \frac{|E_0|}{|B_0|} = \frac{k}{\omega} = c \]
E.M. Wave Summary

- $E \perp B$ and perpendicular to direction of propagation
- Transverse waves
- Speed of propagation $v = c = \lambda f$
- $|E|/|B| = c$
- E.M. waves travel without medium
Typical E.M. wavelength

- **FM Radio:**
  - $f \sim 100 \text{ MHz}$
  - $\lambda = \frac{c}{f} \sim 3\text{ m}$
  - Antenna $\sim O(\text{m})$

- **Cell phone**
  - Antenna $\sim O(0.1\text{ m})$
  - $f = \frac{c}{\lambda} = 3\text{ GHz}$
Energy in E.M. Waves

• Remember:
  - Energy/Volume given by $\frac{1}{2} \varepsilon_0 E^2$ and $\frac{1}{2} \frac{B^2}{\mu_0}$

• Energy density for E.M. wave:
  $$u = \varepsilon_0 E^2$$

• What about power?
Energy in E.M. Waves

\[ \Delta z = c \Delta t \]

- Power/Unit Area (instantaneous)
  \[ \frac{P}{A} = \frac{1}{\mu_0} E B \]
- Power/Unit Area (average)
  \[ \frac{P}{A} = \frac{1}{(2\mu_0)} E_0 B_0 \]
Electromagnetic Waves

• Is light an electromagnetic wave?
  – Check speed and see if we can predict that