Welcome back to 8.033!

Leonhard Euler, Swiss, 1707-1783
Summary of course so far: See study guide

Main focus: be able to solve problems that involve converting between different inertial frames
Today: Geodesics, calculus of variations

• The Euler-Lagrange equation
• Deriving it
• Using it:
  - metrics, Euclidean space geodesics
  - Minkowski space geodesics
  - gravitational redshift
  - brachistochrone problem
  - catenary
Photograph of a brachistochrone experiment. Image removed due to copyright restrictions.
Brachistochrone flicks
Figure by MIT OCW.
Metrics and geodesics

- In an $n$-dimensional space, the *metric* is a (usually position-dependent) $n \times n$ symmetric matrix $g$ that defines the way distances are measured. The length of a curve is $\int d\sigma$, where

$$d\sigma^2 = dr^t \ g \ dr,$$

and $r$ are whatever coordinates you’re using in the space. If you change coordinates, the metric is transformed so that $d\sigma$ stays the same ($d\sigma$ is invariant under all coordinate transformations).

- **Example:** 2D Euclidean space in Cartesian coordinates.

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$d\sigma^2 = dr^t \ g \ dr = \begin{pmatrix} dx & dy \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = dx^2 + dy^2,$$

$$\int d\sigma = \int \sqrt{dr^t \ g \ dr} = \sqrt{dx^2 + dy^2} = \sqrt{1 + (y'(x))^2} \ dx.$$

Applying the Euler-Lagrange equation to this shows that the shortest path between any two points is a straight line.
• **Example:** 4D Minkowski space in Cartesian coordinates \((c = 1\) for simplicity)

\[
g = \eta = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

\[
d\tau^2 = -d\sigma^2 = dx^t \ g \ dx =
\]

\[
= (dx \ dy \ dz \ dt) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \begin{pmatrix}
dx \\
dy \\
dz \\
dt
\end{pmatrix}
\]

\[
= dt^2 - dx^2 - dy^2 - dz^2,
\]

\[
\Delta \tau = \int d\tau = \int \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = \int \sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} \ dt
\]

\[
= \int \sqrt{1 - u^2} \ dt = \int \frac{dt}{\gamma}.
\]

Applying the Euler-Lagrange equation to this shows that the extremal interval between any two events is a straight line though spacetime.