Topics

- Lorentz transformations toolbox
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  - inverse
  - composition ($v$ addition)
  - boosts as rotations
  - the invariant
  - wave 4-vector
  - velocity 4-vector
  - aberration
  - Doppler effect
  - proper time under acceleration
  - calculus of variations
  - metrics, geodesics

- Implications
  - Time dilation
  - Relativity of simultaneity, non-syncronization
  - Length contraction
  - $c$ as universal speed limit
  - Rest length, proper time
Formula summary: transformation toolbox

- Lorentz transformation:
  \[
  \Lambda(\hat{v}) = \left( \begin{array}{cccc} 
  \gamma & 0 & 0 & -\gamma \beta \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  -\gamma \beta & 0 & 0 & \gamma 
  \end{array} \right),
  \]
  i.e.,
  \[
  \begin{pmatrix}
  x' \\
  y' \\
  z' \\
  ct'
  \end{pmatrix} = \begin{pmatrix}
  \gamma(x - \beta ct) \\
  y \\
  z \\
  \gamma(ct - \beta x)
  \end{pmatrix}.
  \]

- This implies all the equations below, derived on the following pages:

  - Inverse Lorentz transformation:
    \[
    \Lambda^{-1}(v) = \Lambda(-v)
    \]

  - Addition of parallel velocities:
    \[
    \Lambda(v_1)\Lambda(v_2) = \Lambda\left(\frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}\right)
    \]

  - Addition of arbitrary velocities:
    \[
    \begin{align*}
    u_x &= \frac{u_x' + v}{1 + \frac{v u_x'}{c^2}} \\
    u_y &= \frac{u_y'\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v u_x'}{c^2}} \\
    u_z &= \frac{u_z'\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v u_x'}{c^2}}
    \end{align*}
    \]

  - Boosts as generalized rotations:
    \[
    \Lambda(-v) = \left( \begin{array}{cccc} 
    \cosh \eta & 0 & 0 & \sinh \eta \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    \sinh \eta & 0 & 0 & \cosh \eta 
    \end{array} \right),
    \]
    where \( \eta \equiv \tanh^{-1} \beta \)

  - All Lorentz matrices \( \Lambda \) satisfy
    \[
    \Lambda^t \eta \Lambda = \eta,
    \]
    where the Minkowski metric is
    \[
    \eta = \left( \begin{array}{cccc} 
    -1 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0 \\
    0 & 0 & -1 & 0 \\
    0 & 0 & 0 & 1 
    \end{array} \right),
    \]
- All Lorentz transforms leave the interval
  \[ \Delta s^2 \equiv \Delta x' \eta \Delta x = \Delta x^2 + \Delta y^2 + \Delta z^2 - (c \Delta t)^2 \]
  invariant
- Wave 4-vector
  \[ \mathbf{K} \equiv \gamma_u \begin{pmatrix} k_x \\ k_y \\ k_z \\ w/c \end{pmatrix}, \]
- Velocity 4-vector
  \[ \mathbf{U} \equiv \gamma_u \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix}, \quad \gamma_u \equiv \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \]
- Aberration:
  \[ \cos \theta' = \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \]
- Doppler effect:
  \[ \omega' = \omega \gamma (1 - \beta \cos \theta) \]

**Formula summary: other**
- Proper time interval:
  \[ \Delta \tau = \int_{t_A}^{t_B} \sqrt{1 - \frac{\dot{r}(t)^2}{c^2}} \, dt \]
- Euler-Lagrange equation:
  \[ \frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} = 0 \]
Implications: time dilation

• In the frame $S$, a clock is at rest at the origin ticking at time intervals that are $\Delta t = 1$ seconds long, so the two consecutive ticks at $t = 0$ and $t = \Delta t$ have coordinates

$$
x_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c\Delta t \end{pmatrix}.
$$

• In the frame $S'$, the coordinates are

$$
x'_1 = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \gamma \Delta t \end{pmatrix},
$$

$$
x'_2 = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ c\Delta t \end{pmatrix} = \begin{pmatrix} -\gamma v\Delta t \\ 0 \\ 0 \\ \gamma c\Delta t \end{pmatrix}.
$$

• So in $S'$, the clock appears to tick at intervals $\Delta t' = \gamma \Delta t > \Delta t$, i.e., slower! (Draw Minkowski diagram.)
Time dilation, cont’d

- The light clock movie says it all:
  \url{http://www.anu.edu.au/Physics/qt/}
- Cosmic ray muon puzzle
  - Created about 10km above ground
  - Half life $1.56 \times 10^{-6}$ second
  - In this time, light travels 0.47 km
  - So how can they reach the ground?
    - $v \approx 0.99c$ gives $\gamma \approx 7$
    - $v \approx 0.9999c$ gives $\gamma \approx 71$
- Leads to twin paradox
Consider two frames in relative motion. For $t = 0$, the Lorentz transformation gives $x' = \gamma x$, where $\gamma > 1$.

**Question:** How long does a yard stick at rest in the unprimed frame look in the primed frame?

1. Longer than one yard
2. Shorter than one yard
3. One yard

**Implications: relativity of simultaneity**

- Consider two events simultaneous in frame $S$:

  \[
  x_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} L \\ 0 \\ 0 \\ 0 \end{pmatrix}.
  \]

- In the frame $S'$, they are

  \[
  x'_1 = \begin{pmatrix} \gamma & 0 & 0 & -\gamma \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma \beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
  \]

  \[
  x'_2 = \begin{pmatrix} \gamma & 0 & 0 & -\gamma \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma \beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} L \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma L \\ 0 \\ 0 \\ -\gamma \beta L \end{pmatrix}.
  \]

- So in $S'$, the second event happened first!

- So $S$-clocks appear unsynchronized in $S'$ - those with larger $x$ run further ahead
Implications: length contraction

- Trickier than time dilation, opposite result (interval appears shorter, not longer)
- In the frame $S$, a yardstick of length $L$ is at rest along the $x$-axis with its endpoints tracing out world lines with coordinates

$$x_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct \end{pmatrix}, \quad x_2 = \begin{pmatrix} L \\ 0 \\ 0 \\ ct \end{pmatrix}.$$  

- In the frame $S'$, these world lines are

$$x'_1 = \begin{pmatrix} x'_1 \\ y'_1 \\ z'_1 \\ ct'_1 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma \beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct \end{pmatrix} = \begin{pmatrix} -\gamma \beta ct \\ 0 \\ 0 \\ ct \end{pmatrix}$$

$$x'_2 = \begin{pmatrix} x'_2 \\ y'_2 \\ z'_2 \\ ct'_2 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma \beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} L \\ 0 \\ 0 \\ ct \end{pmatrix} = \begin{pmatrix} \gamma L - \gamma \beta ct \\ 0 \\ 0 \\ \gamma ct - \gamma \beta L \end{pmatrix}$$

- An observer in $S'$ measures length as $x'_2 - x'_1$ at the same time $t'$, not at the same time $t$.

- Let’s measure at $t' = 0$.
- $t'_1 = 0$ when $t = 0$ — at this time, $x'_1 = 0$
- $t'_2 = 0$ when $ct = \beta L$ - at this time, $x'_2 = \gamma L - \gamma \beta^2 L = L/\gamma$
- So in $S'$-frame, measured length is $L' = L/\gamma$, i.e., shorter

- Let’s work out the new world lines of the yard stick endpoints
- $x'_1 + \beta ct'_1 = 0$, so left endpoint world line is

$$x'_1 = -vt'_1$$

- $x'_2 - \gamma L + \beta (ct'_2 + \gamma \beta L) = 0$, so right endpoint world line is

$$x'_2 = \gamma L - \beta (ct'_2 + \gamma \beta L) = \frac{L}{\gamma} - vt'_2$$

- Length in $S'$ is

$$x'_2 - x'_1 = \frac{L}{\gamma} + v(t'_1 - t'_2) = \frac{L}{\gamma}$$

since both endpoints measured at same time ($t'_1 = t'_2$)

- Draw Minkowski diagram of this
Superluminal communication?

- Velocity addition formula shows that it’s impossible to accelerate something past the speed of light.
- But could there be another way, say a type of radiation that moves faster than light?
- Can an event A influence another event B at spacelike separation (hence transmitting information faster than the speed of light)?
- There is another frame where B happened before A! (PS3)
- Draw Minkowski diagram of this.
- By inertial frame invariance, B can then send a signal that arrives back to A before she sent her initial signal, telling her not to send it.
- Implication: c isn’t merely the speed of light, but the limiting speed for anything.

“Everything is relative” — or is it?

- All observers agree on rest length.
- All observers agree on proper time.
- All observers (as we’ll see later) agree on rest mass.
Transformation toolbox: the inverse Lorentz transform

- Since \( x' = \Lambda(v)x \) and \( x = \Lambda(-v)x' \), we get the consistency requirement
  \[
  x = \Lambda(-v)x' = \Lambda(-v)\Lambda(v)x
  \]
  for any event \( x \), so we must have \( \Lambda(-v) = \Lambda(v)^{-1} \), the matrix inverse of \( \Lambda(v) \).
- Is it?

\[
\Lambda(-v)\Lambda(v) = \begin{pmatrix} \gamma & 0 & 0 & \gamma \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma \beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma \beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]
  i.e., yes!

Transformation toolbox: velocity addition

- If the frame \( S' \) has velocity \( v_1 \) relative to \( S \) and the frame \( S'' \) has velocity \( v_2 \) relative to \( S' \) (both in the x-direction), then what is the speed \( v_3 \) of \( S'' \) relative to \( S \)?
- \( x' = \Lambda(v_1)x \) and \( x'' = \Lambda(v_2)x' = \Lambda(v_2)\Lambda(v_1)x \), so

\[
\Lambda(v_3) = \Lambda(v_2)\Lambda(v_1), \text{ i.e.}
\]

\[
\begin{pmatrix} \gamma_3 & 0 & 0 & -\gamma_3 \beta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_3 \beta_3 & 0 & 0 & \gamma_3 \end{pmatrix} = \begin{pmatrix} \gamma_2 & 0 & 0 & -\gamma_2 \beta_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_2 \beta_2 & 0 & 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 & 0 & -\gamma_1 \beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_1 \beta_1 & 0 & 0 & \gamma_1 \end{pmatrix}
\]

\[
= \gamma_1 \gamma_2 \begin{pmatrix} 1 + \beta_1 \beta_2 & 0 & 0 & -[\beta_1 + \beta_2] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -[\beta_1 + \beta_2] & 0 & 0 & 1 + \beta_1 \beta_2 \end{pmatrix}
\]

- Take ratio between (1,4) and (1,1) elements:

\[
\beta_3 = -\frac{\Lambda(v_3)_{41}}{\Lambda(v_3)_{11}} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}.
\]

- In other words, \( v_3 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \).
Transformation toolbox: perpendicular velocity addition

- Here’s an alternative derivation of velocity addition that easily gives the non-parallel components too (but 4-vector method on next page is simpler)

- If the frame $S'$ has velocity $v$ in the $x$-direction relative to $S$ and a particle has velocity $\mathbf{u}' = (u'_x, u'_y, u'_z)$ in $S'$, then what is its velocity $\mathbf{u}$ in $S$?

- Applying the inverse Lorentz transformation

$$
\begin{align*}
x &= \gamma(x' + vt') \\
y &= y' \\
z &= z' \\
t &= \gamma(t' + vx'/c^2)
\end{align*}
$$

to two nearby points on the particle’s world line and subtracting gives

$$
\begin{align*}
dx &= \gamma(dx' + vdt') \\
dy &= dy' \\
dz &= dz' \\
dt &= \gamma(dt' + vdx'/c^2).
\end{align*}
$$

- Answer:

$$
\begin{align*}
u_x &= \frac{dx}{dt} = \frac{\gamma(dx' + vdt')}{\gamma(dt' + vdx'/c^2)} = \frac{\frac{dx'}{dt} + \frac{v}{c^2} \frac{dx'}{dt}}{1 + \frac{v}{c^2} \frac{dx'}{dt}} = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \\
u_y &= \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + vdx'/c^2)} = \frac{\gamma^{-1} \frac{dy'}{dt} + \frac{v}{c^2} \frac{dx'}{dt}}{1 + \frac{v}{c^2} \frac{dx'}{dt}} = \frac{u'_y \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_y v}{c^2}} \\
u_z &= \frac{dz}{dt} = \frac{dz'}{\gamma(dt' + vdx'/c^2)} = \frac{\gamma^{-1} \frac{dz'}{dt} + \frac{v}{c^2} \frac{dx'}{dt}}{1 + \frac{v}{c^2} \frac{dx'}{dt}} = \frac{u'_z \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_z v}{c^2}}
\end{align*}
$$
Transformation toolbox: velocity as a 4-vector

- For a particle moving along its world-line, define its velocity 4-vector

\[ U \equiv \frac{dX}{d\tau} = \gamma_u \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix}, \]

where

\[ \gamma_u \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

- This is the derivative of its 4-vector x w.r.t. its proper time \( \tau \), since \( d\tau = dt/\gamma_u \)

- \( U' = \Lambda U \):

\[ U' = \frac{dX'}{d\tau'} = \Lambda \frac{dX}{d\tau} = \Lambda U, \]

since the proper time interval \( d\tau \) is Lorentz-invariant

- This means that all velocity 4-vectors are normalized so that

\[ U'\eta U = -c^2. \]

- This immediately gives the velocity addition formulas:

\[ U' = \gamma_u' \begin{pmatrix} u_x' \\ u_y' \\ u_z' \\ c \end{pmatrix} = \Lambda(-v)U = \gamma_u \begin{pmatrix} \gamma & 0 & 0 & \gamma \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma \beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix} \]

\[ = \gamma_u' \begin{pmatrix} \gamma_u \gamma [u_x + v] \\ \gamma_u u_y \\ \gamma_u u_z [1 + \frac{u_x v}{c^2}] \\ \gamma_u \gamma [1 + \frac{u_x v}{c^2}] c \end{pmatrix}, \]

where \( \gamma_u' = \gamma_u \gamma \left[1 + \frac{u_x v}{c^2}\right] \) — this last equation follows from the fact that the 4-vector normalization in Lorentz invariant, i.e., \( u'^\eta u' = u'^\eta u = -1. \)

- The 1st 3 components give the velocity addition equations we derived previously.
Transformation toolbox:
boosts as generalized rotations

- A “boost” is a Lorentz transformation with no rotation
- A rotation around the z-axis by angle θ is given by the transformation
  \[
  \begin{pmatrix}
  \cos \theta & \sin \theta & 0 & 0 \\
  -\sin \theta & \cos \theta & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
  \end{pmatrix}
  \]

- We can think of a boost in the x-direction as a rotation by an imaginary angle in the \((x, ct)\)-plane:
  \[
  \Lambda(-v) = \begin{pmatrix}
  \gamma & 0 & 0 & \gamma \beta \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  \gamma \beta & 0 & 0 & \gamma
  \end{pmatrix} = \begin{pmatrix}
  \cosh \eta & 0 & 0 & \sinh \eta \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  \sinh \eta & 0 & 0 & \cosh \eta
  \end{pmatrix},
  \]
  where \(\eta \equiv \tanh^{-1} \beta\) is called the rapidity.

- Proof: use hyperbolic trig identities on next page
- Implication: for multiple boosts in same direction, rapidities add and hence the order doesn’t matter

Hyperbolic trig reminders

\[
\begin{align*}
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
\cosh^{-1} x &= \ln(x + \sqrt{x^2 - 1}) \\
\sinh^{-1} x &= \ln(x + \sqrt{x^2 + 1}) \\
\tanh^{-1} x &= \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right)
\end{align*}
\]

\[
\begin{align*}
\cosh \tanh^{-1} x &= \frac{1}{\sqrt{1-x^2}} \\
\sinh \tanh^{-1} x &= \frac{x}{\sqrt{1-x^2}} \\
\cosh^2 x - \sinh^2 x &= 1
\end{align*}
\]
The Lorentz invariant

- The Minkowski metric
  \[
  \eta = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & -1
  \end{pmatrix}
  \]

  is left invariant by all Lorentz matrices \( \Lambda \):
  \[
  \Lambda^t \eta \Lambda = \eta
  \]
  (indeed, this equation is often used to define the set of Lorentz matrices — for comparison, \( \Lambda^t I \Lambda = I \) would define rotation matrices)

- Proof: Show that works for boost along \( x \)-axis. Show that works for rotation along \( y \)-axis or \( z \)-axis. General case is equivalent to applying such transformations in succession.

- All Lorentz transforms leave the quantity
  \[
  x^t \eta x = x^2 + y^2 + z^2 - (ct)^2
  \]
  invariant

- Proof:
  \[
  x'^t \eta x' = (\Lambda x)^t \eta (\Lambda x) = x^t (\Lambda^t \eta \Lambda)x = x^t \eta x
  \]
  (More generally, the same calculation shows that \( x^t \eta y \) is invariant)

- So just as the usual Euclidean squared length \( |r|^2 = r \cdot r = r^t r = r^t i r \) of a 3-vector is rotationally invariant, the generalized “length” \( x^t \eta x \) of a 4-vector is Lorentz-invariant.

- It can be positive or negative

- For events \( x_1 \) and \( x_2 \), their Lorentz-invariant separation is defined as
  \[
  \Delta \sigma^2 \equiv \Delta x^t \eta \Delta x = \Delta x^2 + \Delta y^2 + \Delta z^2 - (c \Delta t)^2
  \]

- A separation \( \Delta \sigma^2 = 0 \) is called null

- A separation \( \Delta \sigma^2 > 0 \) is called spacelike, and
  \[
  \Delta \sigma \equiv \sqrt{\Delta \sigma^2}
  \]
  is called the proper distance (the distance measured in a frame where the events are simultaneous)

- A separation \( \Delta \sigma^2 < 0 \) is called timelike, and
  \[
  \Delta \tau \equiv \sqrt{-\Delta \sigma^2}
  \]
  is called the proper time interval (the time interval measured in a frame where the events are at the same place)

- More generally, any 4-vector is either null, spacelike of timelike.

- The velocity 4-vector \( U \) is always timelike.
Transforming a wave vector

- A plane wave

\[ E(x) = \sin(k_xx + k_yy + k_zz - \omega t) \]  

(1)

is defined by the four numbers

\[ K \equiv \begin{pmatrix} k_x \\ k_y \\ k_z \\ \omega/c \end{pmatrix}. \]

- If the wave propagates with the speed of light \( c \) (like for an electromagnetic or gravitational wave), then the frequency is determined by the 3D wave vector \((k_x, k_y, k_z)\) through the relation \( \omega/c = k \), where \( k \equiv \sqrt{k_x^2 + k_y^2 + k_z^2} \).

- How does the 4-vector \( K \) transform under Lorentz transformations? Let’s see.

- Using the Minkowski matrix, we can rewrite equation (1) as

\[ E(X) = \sin(K' \eta X). \]

- Let’s Lorentz transform this: \( X \rightarrow X', \ K \rightarrow K' \). Using that \( X' = \Lambda X \), let’s determine \( K' \).

\[ E' = \sin(K' \eta X') = \sin(K' \eta \Lambda X) = \sin[(\Lambda^{-1}K')' \eta \Lambda X] = \sin[(\Lambda^{-1}K')' \eta X]. \]

- This equals \( E \) if \( \Lambda^{-1}K' = K \), i.e., if the wave 4-vector transforms just as a normal 4-vector:

\[ K' = \Lambda K \]

- This argument assumed that \( E' = E \). Later we’ll see that the electric and magnetic fields do in fact change under Lorentz transforms, but not in a way that spoils the above derivation (in short, the phase of the wave, \( K' \eta X \), must be Lorentz invariant).

- So a plane wave \( K \) in \( S \) is also a plane wave in \( S' \), and the wave 4-vector transforms in exactly the same way as \( X \) does.
Aberration and Doppler effects

- Consider a plane wave propagating with speed $c$ in the frame $S$:

$$K = k \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \\ 1 \end{pmatrix},$$

where $ck$ is the wave frequency and the angles $\theta$ and $\phi$ give the propagation direction in polar coordinates.

- Let’s Lorentz transform this into a frame $S'$ moving with speed $v$ relative to $S$ in the $z$-direction: $k' = \Lambda k$, i.e.,

$$K' = k' \begin{pmatrix} \sin \theta' \cos \phi' \\ \sin \theta' \sin \phi' \\ \cos \theta' \\ 1 \end{pmatrix} = k \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\gamma \beta \\ 0 & 0 & -\gamma \beta & \gamma \end{pmatrix} \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \\ 1 \end{pmatrix}$$

so

$$\phi' = \phi$$
$$\cos \theta' = \frac{\cos \theta - \beta}{1 - \beta \cos \theta}$$
$$k' = k \gamma (1 - \beta \cos \theta)$$

- This matches equations (1)-(4) in the Weiskopf et al ray tracing handout.

- The change in the angle $\theta$ is known as aberration.

- The change in frequency $ck$ is known as the Doppler shift — note that since $k = 2\pi/\lambda$, we have $\lambda' / \lambda = k / k'$.

- If we instead take the ratio $\sqrt{k'_x^2 + k'_y^2 / k'_z^2}$ above, we obtain the mathematically equivalent form of the aberration formula given by Resnick (2-27b):

$$\tan \theta' = \frac{\sin \theta}{\gamma (\cos \theta - \beta)}$$
• Examine classical limits

• Transverse Doppler effect: \( \cos \theta = 0 \) gives \( \omega' = \omega \gamma, \) i.e., simple time dilation (classically, \( \omega' = \omega, \) i.e., no transverse effect)

• Longitudinal doppler effect: \( \cos \theta = 1 \) gives

\[
\frac{\omega'}{\omega} = \gamma(1 - \beta) = \sqrt{\frac{1 - \beta}{1 + \beta}}
\]

• For comparison, classical physics, moving observer:

\[
\frac{\omega'}{\omega} = 1 - \beta.
\]

• For comparison, classical physics, moving source:

\[
\frac{\omega'}{\omega} = \frac{1}{1 + \beta}
\]
Accelerated motion & proper time

- Consider a clock moving along a curve \( \mathbf{r}(t) \) through spacetime, as measured in a frame \( S \). During an infinitesimal time interval between \( t \) and \( t + dt \), it moves with velocity \( \mathbf{u}(t) = \dot{\mathbf{r}}(t) \) and measures a proper time interval

\[
d\tau = \frac{dt}{\gamma_u} = \sqrt{1 - \frac{[\dot{\mathbf{r}}(t)]^2}{c^2}} dt.
\]

- The proper time interval (a.k.a. wristwatch time) measured by the clock as it moves from event A to event B along this path is

\[
\Delta\tau = \int_{t_A}^{t_B} d\tau = \int_{t_A}^{t_B} \sqrt{1 - \frac{[\dot{\mathbf{r}}(t)]^2}{c^2}} dt
\]

- If the two events are at the same position in \( S \), i.e., if \( \mathbf{r}(t_A) = \mathbf{r}(t_B) \), then the path \( \mathbf{r}(t) \) between the two events that maximizes \( \Delta\tau \) is clearly the straight line \( \mathbf{r}(t) = \mathbf{r}(t_A) \) where the clock never moves, giving \( \mathbf{u} = 0 \) and \( \Delta\tau = \Delta t = t_B - t_A \).

- For any two events with timelike separation, the proper time is again maximized when the path between the two points is a straight line through spacetime.

Proof: Lorentz transform to a frame \( S' \) where \( A \) and \( B \) are at the same position, conclude the path is a straight line in \( S' \) and use the fact that the Lorentz transform of a straight line through spacetime is always a straight line through spacetime.

- One can also deduce this with calculus of variations, which is overkill for this simple case.

Calculus of variations

- The much more general optimization problem of finding the path \( x(t) \) that minimizes or maximizes a quantity

\[
S[x] = \int_{t_0}^{t_1} f[t, x(t), \dot{x}(t)] dt
\]

subject to the constraints that \( x(t_0) = x_0 \) and \( x(t_1) = x_1 \) reduces to solving the differential equation known as the Euler-Lagrange equation:

\[
\frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} = 0.
\]

- Here the meaning of \( \frac{\partial f}{\partial x} \) is simply the partial derivative of \( f \) with respect to its third argument, i.e., just treat \( \dot{x} \) as a variable totally independent of \( x \) when evaluating this derivative.
Metrics and geodesics

- In an \( n \)-dimensional space, the metric is a (usually position-dependent) \( n \times n \) symmetric matrix \( g \) that defines the way distances are measured. The length of a curve is \( \int d\sigma \), where

\[
d\sigma^2 = dr^i \ g \ dr^i,
\]

and \( r \) are whatever coordinates you’re using in the space. If you change coordinates, the metric is transformed so that \( d\sigma \) stays the same (\( d\sigma \) is invariant under all coordinate transformations).

- **Example:** 2D Euclidean space in Cartesian coordinates.

\[
g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
d\sigma^2 = dr^i \ g \ dr^i = (dx \ dy) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = dx^2 + dy^2,
\]

\[
\int d\sigma = \int \sqrt{dr^i \ g \ dr^i} = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'(x)^2}dx.
\]

Applying the Euler-Lagrange equation to this shows that the shortest path between any two points is a straight line.

- **Example:** 4D Minkowski space in Cartesian coordinates (\( c = 1 \) for simplicity)

\[
g = \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
\]

\[
d\tau^2 = -d\sigma^2 = dx^i \ g \ dx^i =
\]

\[
= (dx \ dy \ dz \ dt) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix}
\]

\[
= dt^2 - dx^2 - xy^2 - dz^2,
\]

\[
\Delta \tau = \int d\tau = \int \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = \int \sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}dt
\]

\[
= \int \sqrt{1 - \dot{\gamma}^2}dt = \int \frac{dt}{\gamma}.
\]

Applying the Euler-Lagrange equation to this shows that the extremal interval between any two events is a straight line through spacetime.